UNIVERSITY OF WARWICK

Ph.D. Thesis

ELEMENTARY CATASTROPHES

by

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C O N T E N T S

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The theory of catastrophe as introduced by Thom [1,2] could well be the most important step in the creation of a reliable method for mathematical modeling of complex situations. A thorough justification for this statement and the discussion of applications that have already been worked out is not within the scope of this thesis. We assume that a good case has been made in terms of mathematical interest and applicability of the results to warrant the closer attention to the details of the mathematical implications of the theory. In line with the general principle that it is easier to describe how to make something than to describe the finished project, we find the simple formula for potential functions have complex geometric consequences. It is perhaps not too much of a distortion to say that this thesis is a description of some geometric properties implied by some simple families of polynomial functions of two variables.

In Chapter I we give a summary of the arguments given by Thom for the derivation of the families of potential functions that give the elementary catastrophes from a fairly general mathematical level. The pre-mathematical level ancestry of these elementary catastrophes and their relations with the generalised catastrophes has been left to reference to Thom's own work. Besides giving the resume we also look at an alternative method of derivation of the formulae for the elementary catastrophes described by Mather (see I, §4). But again we do not go into detail. This time it is the opposite reason: in this case the original material is sparse whereas in the former case there is plenty which needs smelting down.

The parabolic umbilic, one of Thom's list of seven elementary catastrophes is a good example of the simple formula containing a large amount of geometric information. Using an elementary approach we give pictures of the bifurcation set in the unfolding space for the parabolic umbilic in Appendix I (3-dimensional pictures for Thom's parabolic umbilic, I.H.E.S.40 (1971).) The results here have been obtained using a combination of elementary mathematics and computing. The diagrams drawn by computer being confirmed as necessary and sufficient by the analytic work. Further results for the parabolic umbilic obtained by lifting to split the regions of the 3-dimensional pictures into separate critical points are given in Chapter II. Following this we consider the relationship between the structurally stable maps $\mathbb{R}^n \to \mathbb{R}^n$ and the umbilic catastrophes. Also in Chapter II we

take a look at the relationship of the drawings given in Appendix I to some Boardman singularity sets of related singular maps.

The work of Chapter III is an attempt to get a detailed description of a section of the double Riemann-Hugoniot catastrophe. It turns out to be much more difficult than the case of the parabolic umbilic to confirm all the computer results analytically. These results are drawn then without the confirmation that all possible cases have been covered. The results, however, will be a useful step in understanding the geometry of the double Riemann-Hugoniot catastrophe.

In the last chapter we turn our attention to the full double Riemann-Hugoniot catastrophe. The first section takes a preliminary look at the geometry of the function space relating to topological and differential equivalence classes. In trying to get at pictures of the type used in applications of the theory, we can first try to understand the local theory. In the second section of Chapter IV we derive the local topological bifurcation set to give this local picture. From the other end we can consider the general global properties of the potential functions of the family and we do this in the third section of the chapter. In the two remaining sections we give the reasons for the name of the double Riemann-Hugoniot catastrophe and make some suggestions for further investigation.

The computer programs directly quoted in the text are given with rough documentation in Appendix II.

Acknowledgments

I would like to thank my supervisor, Professor E.C. Zeeman, for much encouragement and many fruitful suggestions in the course of the work involved in this thesis. While being entangled with the development of ideas, there have been many people who's interest in these ideas has been most stimulating. I would particularly like to thank the more sceptical with their penetrating questions. The transformation of an untidily written manuscript into a typewritten version has been accomplished by Mrs. Lange to whom I should like to express my thanks.

CHAPTER I - GENERAL THEORY

§1 Introduction

The aim of this chapter is to give some ideas of the author's view of the general theory of elementary catastrophes as introduced by Thom [1] . The general theory of catastrophes has been discussed very fully in Thom's book [1] and we intend only to give enough background to put the remaining chapters in perspective. Also, since the publication of the ideas relating to elementary catastrophes, several details have been clarified by other authors and some note of this will also be given.

Thom has considered the theory of catastrophes from a very fundamental level [1] . He has then made reasonable assumptions to reduce the theory to a mathematical level that is accessible to present techniques. It is this latter level of generalising which we consider and do not look at the deeper levels.

§2 The Mathematical Context

The first step in the construction of the mathematical theory of elementary catastrophes is a general space $\mathcal{O}$ in which observable processes take place. In Thom's work this is usually taken to be space time and hence limited to four dimensions. However, applications considered by Zeeman [2] suggest that this is an unnecessary restriction and so we can assume a general Euclidian space.

The state of the system, a deliberately vague term, at each point $x \in \mathcal{O}$ is determined by a rule with a two stage description. Associated with x there is a vector field $X(x)$ defined on a manifold M. The dominant attractor of this vector field A, then determines the state at x. We call the manifold M the state space, the positions of the attractor A in M defining the state.

To find the state at each point in a region $U \in \mathcal{O}$ we must construct a function $X : U \longrightarrow \mathcal{X}(M)$ (the space of vector fields on M). This function is assumed to be sufficiently smooth and bijective onto the image $X(U)$. The space $\mathcal{X}(M)$ is supposed to be decomposed into equivalence classes related to the attractors of the vector fields and hence to the states in $\mathcal{O}$. The image $X(U)$ is assumed to intersect these equivalence classes in a useful way and the inverse images of the different equivalence classes will relate to the variation in state over U. It is the points of U having all neighbourhoods containing points in distinct states that are called the <u>catastrophe points</u>.

To get the elementary catastrophes Thom first assumes that we only need to consider gradient fields and then makes the further simplification to the case of germs of maps $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. The relationship between structural stability and the gradient fields has been the subject of several specific conjectures by Thom which were discussed by the author in a mainly expository M.Sc. Dissertation [3]. Briefly we can say that nothing has cropped up to make the assumption of gradient field appear too specific to be tenable. Thus in place of $\mathcal{X}(M)$ we have a space of germs of maps and the structure of this can be studied through the use of the theory of singularities of maps $(\mathbb{R}^m, 0)$ $(\mathbb{R}^p, 0)$.

§3 Thom's Elementary Catastrophes

Thom's derivation of the analytic form of the elementary catastrophes appears in his book and we give here a resumé of the line of argument.

The first stage is an algebraic definition of a strictly isolated singularity followed by an argument to show that given a potential function we can find an arbitrarily close approximation with only strictly isolated singularities. The argument depends on transversality and shows that 'most' potential functions have only strictly isolated singularities. We now look at an individual strictly isolated singularity for a map $V : \mathbb{R}^n \to \mathbb{R}$. Using Morse theory, Thom shows that the corank of the quadratic form in the Taylor expansion in the neighbourhood of the singularity is an intrinsic property of the singularity. Considering then the codimension of sets in the jet space $J^2(n,1)$ he shows that we only need to consider cases with the quadratic form with corank 1 and 2. This restriction on corank depends on assumption (i) $X : U \longrightarrow \mathcal{X}(M)$, is transversal to sets defined by pulling back from $J^2(n,1)$; (ii) dimension $\mathcal{O} \leq 5$. Next he takes the important step of justifying the invariance of the geometry in the space of potential functions with respect to variation in n. Thus we know that we only need to consider functions of the form

$$V(x_1, \dots, x_n) = \sum_{s=3}^{\infty} a_s x_1^{\,s} + Q(x_2, \dots, x_n)$$

$$V(x \;, \dots, x_n) = \sum_{s,t \geq 3}^{\infty} a_{st} x_1^{\,s} x_2^{\,t} + Q(x_3, \dots, x_n)$$

where Q( ) are non singular quadratic forms and can be ignored in the relevant geometry of $C^\infty(\mathbb{R}^n, \mathbb{R})$.

The potential functions $V(x_1, \dots x_n)$ that we have obtained have unstable singularities at the origin. As a last step in his derivation

Thom gets the universal unfoldings of these unstable maps expressed as a k-parameter family of potential functions based on V at the origin of parameters or alternatively as a stable map $(\mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R} \times \mathbb{R}^k, 0)$. The parameters of the universal unfolding are then considered as coordinates transverse to the topological equivalence class of V in $C^\infty(\mathbb{R}^n, \mathbb{R})$. This means that these k parameters can be considered as coordinates in U induced by pulling back through the map X. By limiting k to be ≤ 4 we can get the exhaustive list of formulae for the seven elementary catastrophes [1,4].

The development given by Thom for these formulae from the assumption of gradient vector fields is not completely rigorous but no results have yet come to light making the final answers invalid.

§ 4 Mather's Catastrophes

At the University of Warwick Symposium in 1969, Mather [5] announced a result giving a rigorous deduction confirming the formulae for the seven elementary catastrophes. The complete paper has not yet been published nor does it appear to be under consideration for any journal. An elusive manuscript version does exist but the author has been unable to get a copy. We give here a brief summary of the seminar given by Mather at the Symposium.

The proof begins with the set of smooth potential functions on X x M, $C^\infty(X \times M)$ where X and M are both compact. For each map Mather defines the singularity set

$$\Sigma(f) = \{(x,m) \mid f \mid \{x\} \times M \text{ has a critical point at } m\}$$

and shows by transversality that for an open dense set of maps, $U \subset C^\infty(X \times M)$, $\Sigma(f)$ is a submanifold of X x M with $\dim \Sigma(f) = \dim X$. (In Chapter II of this thesis we look at $\Sigma(f)$ relating to the parabolic umbilic.) The next step is to consider the singularities of the projection

$$\pi : \Sigma(f) \to X$$

and Mather has proved that if $\dim X = 4$ then for an open dense $U \subset C^\infty(X \times M)$ there are at most seven types of singularity for π. If $F_1, \ldots, F_7$ are the caronconsole germs then for any $f \in U$, if $\pi : \Sigma(f) \to X$ has a singularity at (x_0, m_0) then $\exists$ a local commutative diagram

$$\begin{tikzcd}
(\Sigma(f), (x_o, m_o)) \arrow[r, "\pi"] \arrow[d, "k_1"] & (X, x) \arrow[d, "k_2"] \\
(\mathbb{R}^4, 0) \arrow[r, "F_i"] & (\mathbb{R}^4, 0)
\end{tikzcd}$$

where F_i is one of the cannonical germs and k_1, k_2 are germs of diffeomorphisms. These germs F_i are then shown to have up to diffeomorphism-equivalence a unique related map $f_i : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ with $\pi(\Sigma(f_i))$ having the particular type of singularity F_i at the origin. The maps f_i have unstable singularities at the origin and from these Mather generates the universal unfoldings, $G_i : (\mathbb{R}^n \times \mathbb{R}^4, 0) \to (\mathbb{R} \times \mathbb{R}^4, 0)$.

The final results of Mather's process are the same seven elementary catastrophes produced by Thom. However, whereas Mather assumes C^∞-functions and uses C^∞- equivalence to get the equivalence classes, Thom aims at C^∞- functions under topological equivalence [4] . In Chapter IV we see that some differences between topological and differentiable cases can occur for higher elementary catastrophes. Even for all the seven elementary catastrophes it has not been proved that the bifurcation sets under the two different types of equivalence on potential functions will be differentiably equivalent.

5 Conclusions

In this chapter we have given a brief indication of the pedigree of the elementary catastrophes and pointed out some of the gaps in their derivation. In the remainder of this thesis we explore details of some of the elementary catastrophes and in Chapter II consider the related singularities of smooth maps. It is clear from this chapter and from specific points in other chapters, that although a substantial start has been made to the mathematical theory of elementary catastrophes, much more remains as yet undone.

Reference

[1]	R. Thom	Stabilite Structurelle et Morphogenese, Benjamin (to appear).
[2]	E.C. Zeeman	Applications of Catastrophes (to appear).
[3]	A.N. Godwin	M.Sc. Dissertation, University of Warwick (1968).
[4]	R. Thom	Topological Models in Biology, Topology 8. (1969) 313-335
[5]	J. Mather	Universal Unfoldings, Preprint of Proceedings of Warwick University Symposium on Dynamical Systems (1969).

CHAPTER II - THE PARABOLIC UMBILIC

In this chapter we consider some geometry related to the seventh of Thom's list of elementary catastrophes, the parabolic umbilic.

§ 1 The Family of Potential Functions

If we consider the potential function

$$V_p : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto x^2 y + y^4$$

and then calculate an unfolding, we get a family of potential functions

$$V_p(u,v,w,t) : \mathbb{R}^2 \longrightarrow \mathbb{R} \; ; \qquad \text{-------------} \quad (1)$$
$$(x,y) \longmapsto x^2 y + y^4 - ux - vy + wx^2 + ty^2 .$$

Examining the critical points of this family we can calculate a bifurcation set B as a subset of $\mathbb{R}^4$. The details of this calculation are given in Appendix I.

In Zeeman's applications of the theory [1], it is useful not only to have a picture of the set B, but also to include some indication of the state variables x,y. For the parabolic umbilic we can do this by considering the equation for the critical points of potential function (1) i.e.

$$2x(y + w) - u = 0 \; ;$$
$$x^2 + 4y^3 + 2ty - v = 0 \; ;$$

in the form

$$x = \frac{u}{2\,(y + w)} \; ; \qquad \text{-------------} \quad (2)$$

$$v = \frac{u^2}{4\,(y + w)^2} + 4y^3 + 2ty \qquad \text{-------------} \quad (3)$$

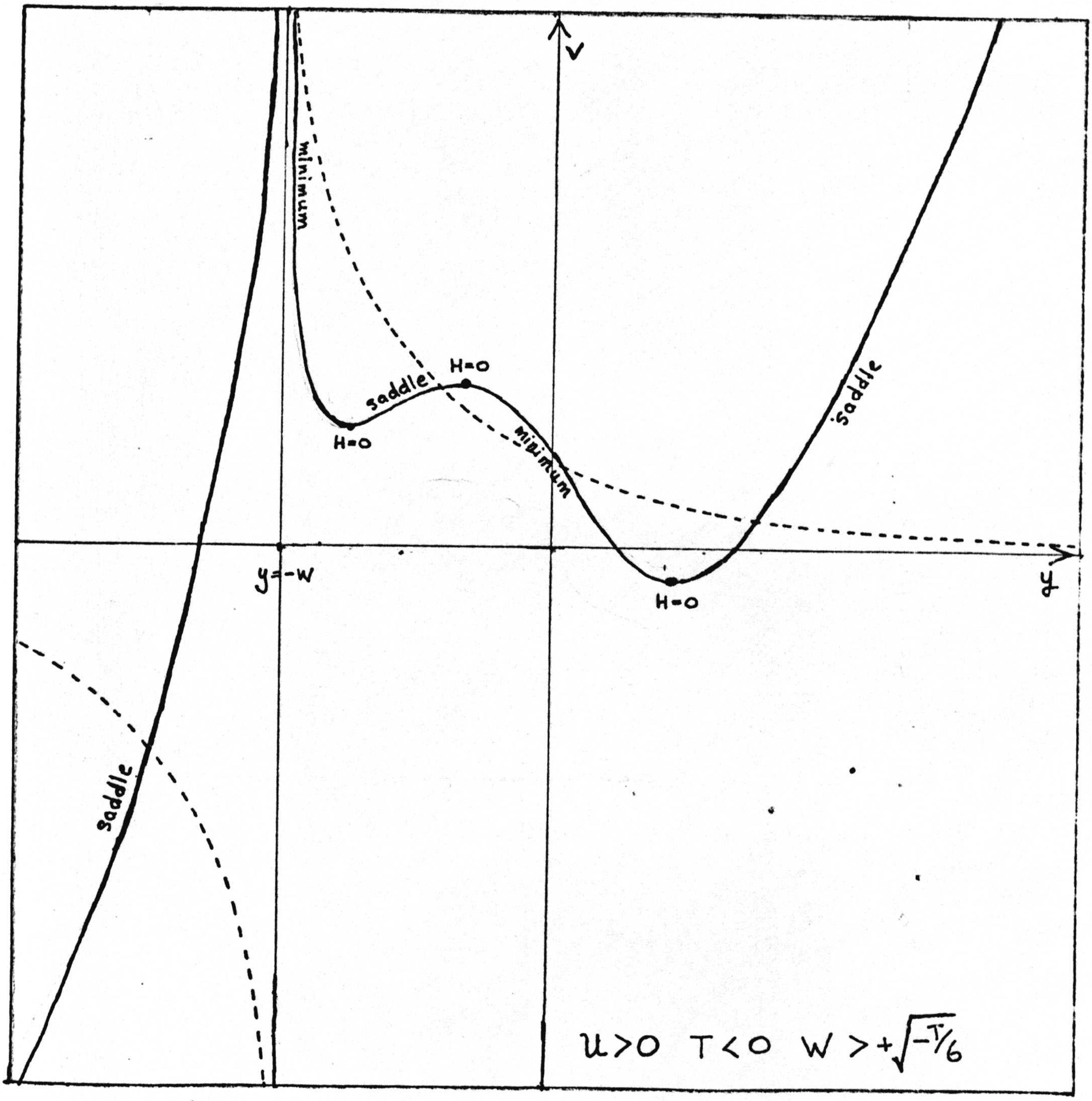
On further consideration of the (y,v) graphs we see that

$$\frac{dv}{dy} = - \frac{u^2}{2\,(y + w)^3} + 12y^2 + 2t = 0$$

corresponds to the vanishing of the Hessian determinant

$$H = \begin{vmatrix} \dfrac{\partial^2 V_p}{\partial x^2} & \dfrac{\partial^2 V_p}{\partial x \partial y} \\ \dfrac{\partial^2 V_p}{\partial x \partial y} & \dfrac{\partial^2 V_p}{\partial y^2} \end{vmatrix} = 4((y + w)(6y^2 + t) - x^2) \quad \text{- -} \quad (4)$$

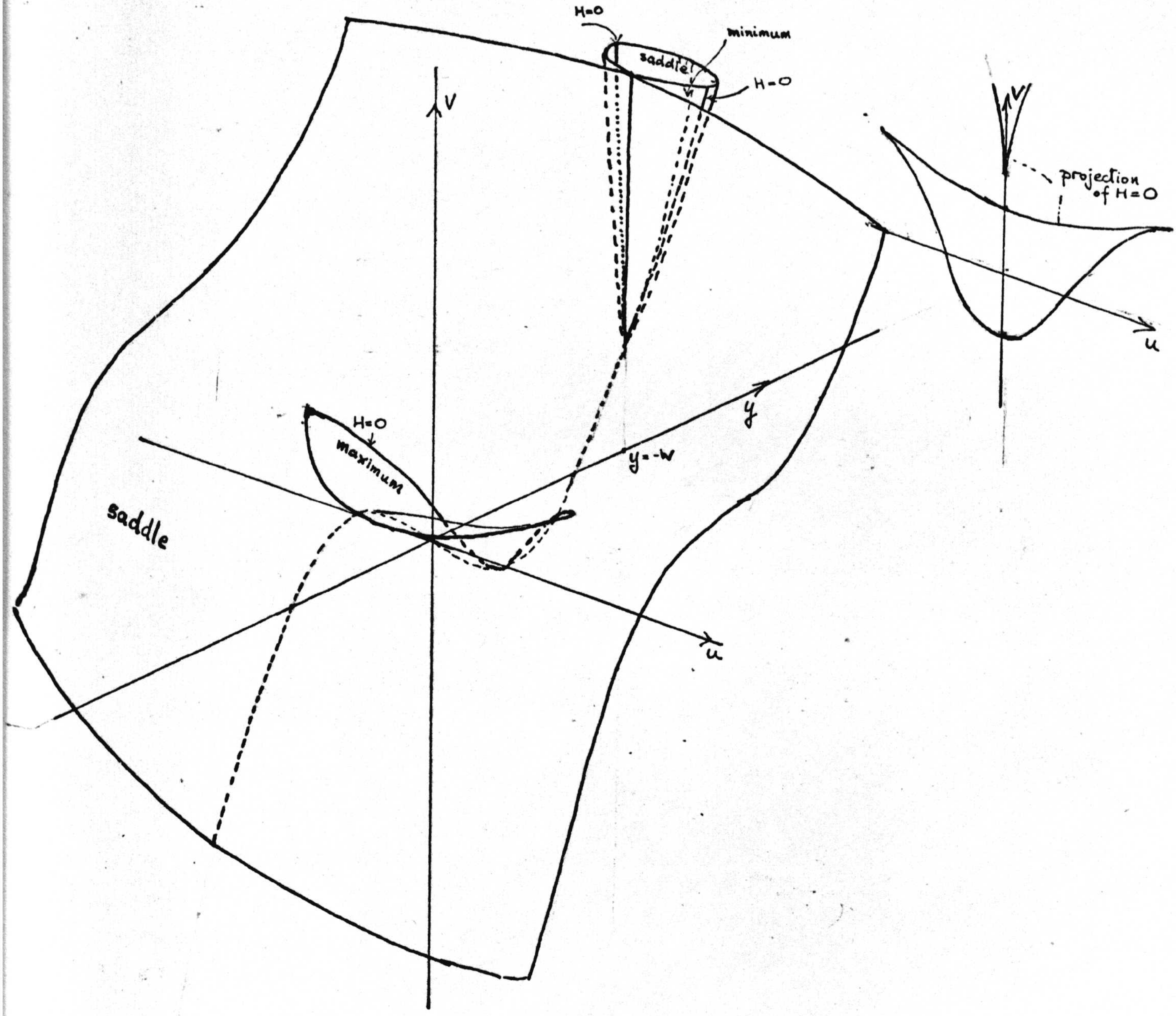
and the monotonic sections of the graph correspond to different types



— $V = \frac{U^2}{4(y+W)^2} + 4y^3 + 2Ty$

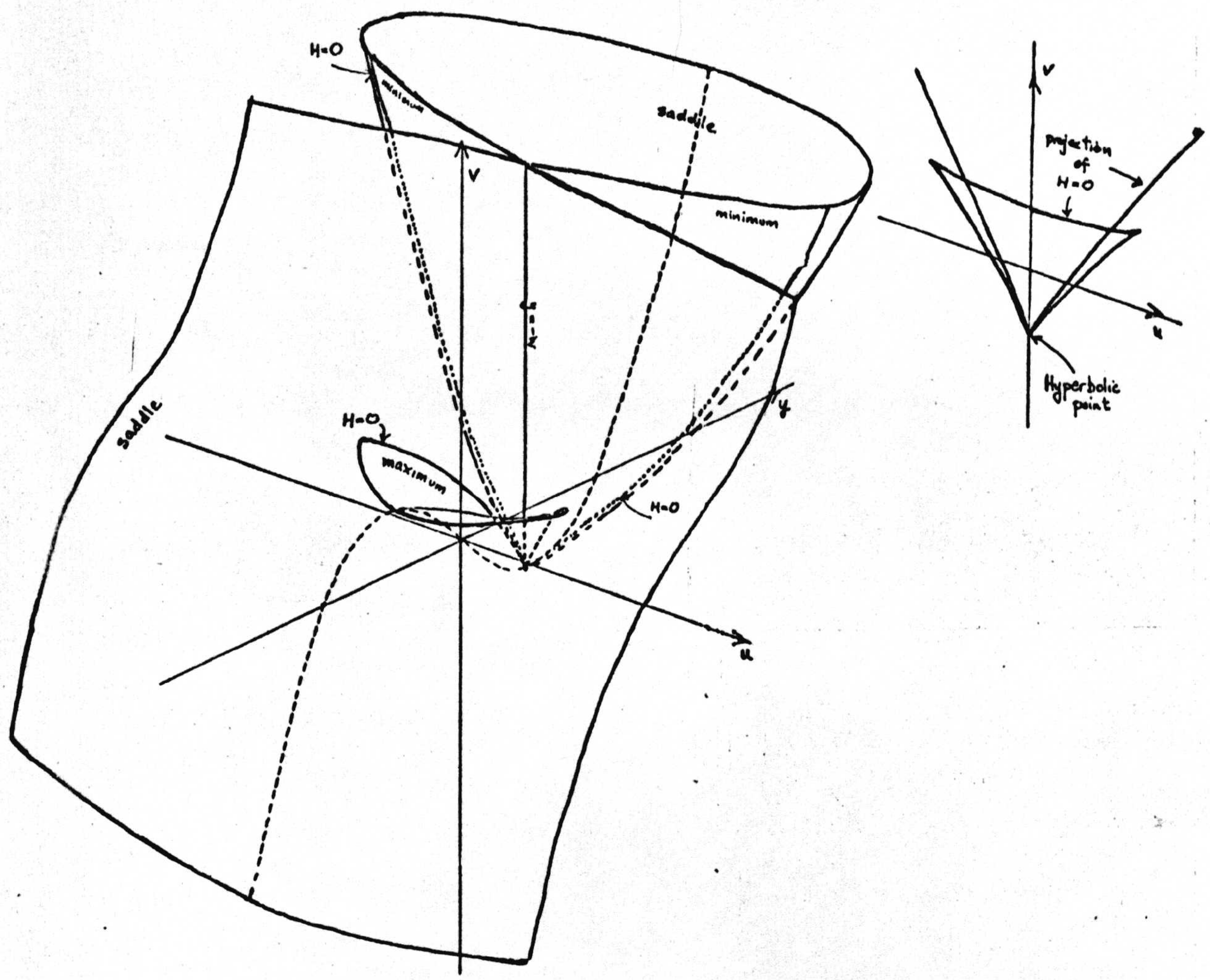
---- $V = \frac{U}{2(y+W)}$.

Fig 1.



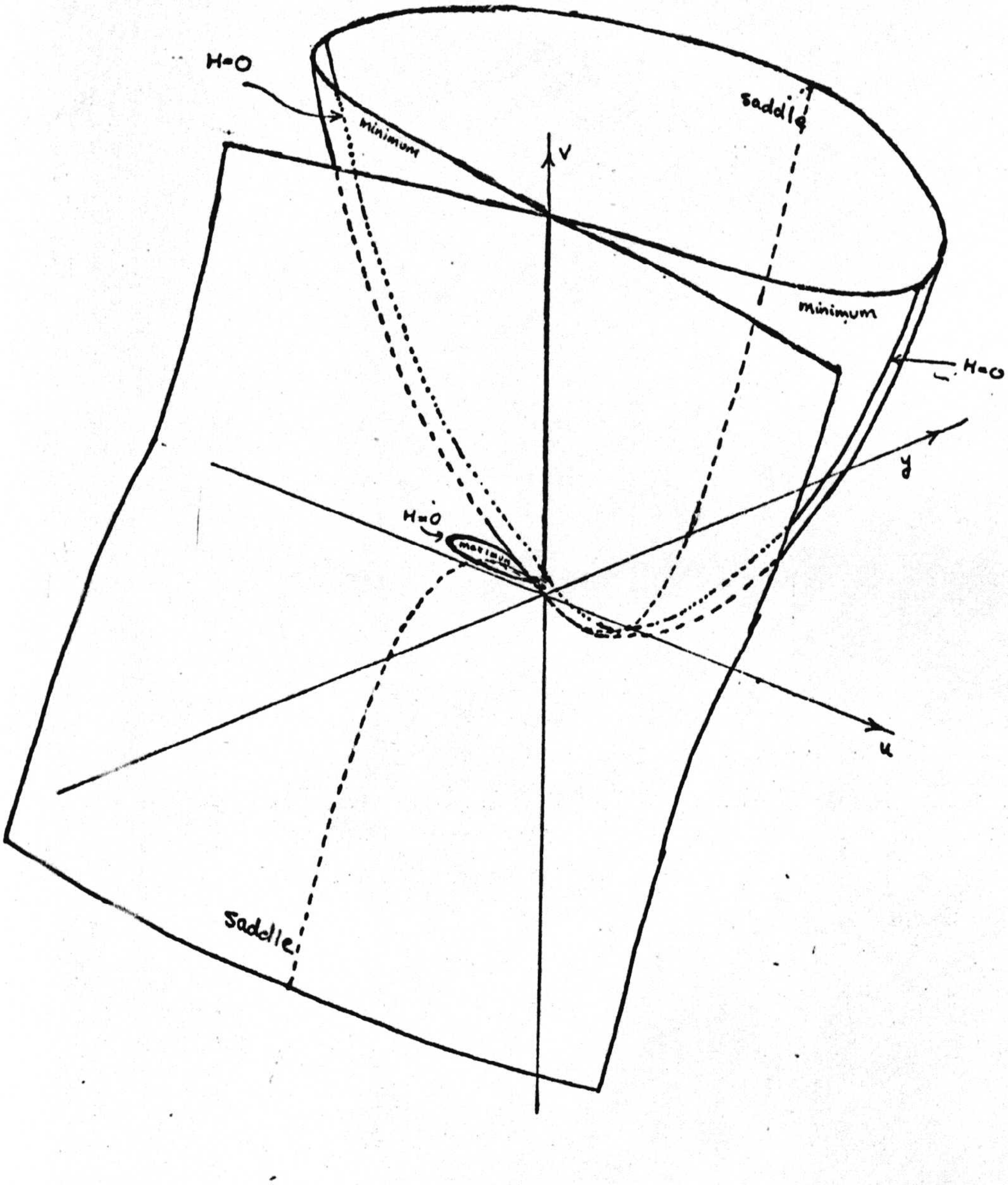
$T<0 \quad W< -2\sqrt{-T/6}$

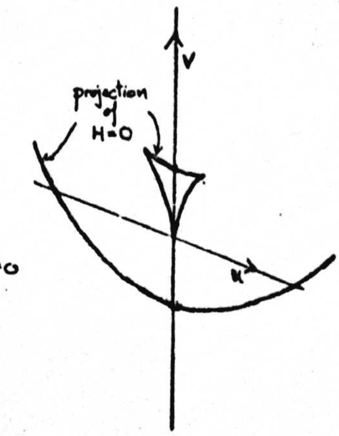
Fig 2.



$$\boxed{T<0 \quad W=-\sqrt{-T/6}}$$

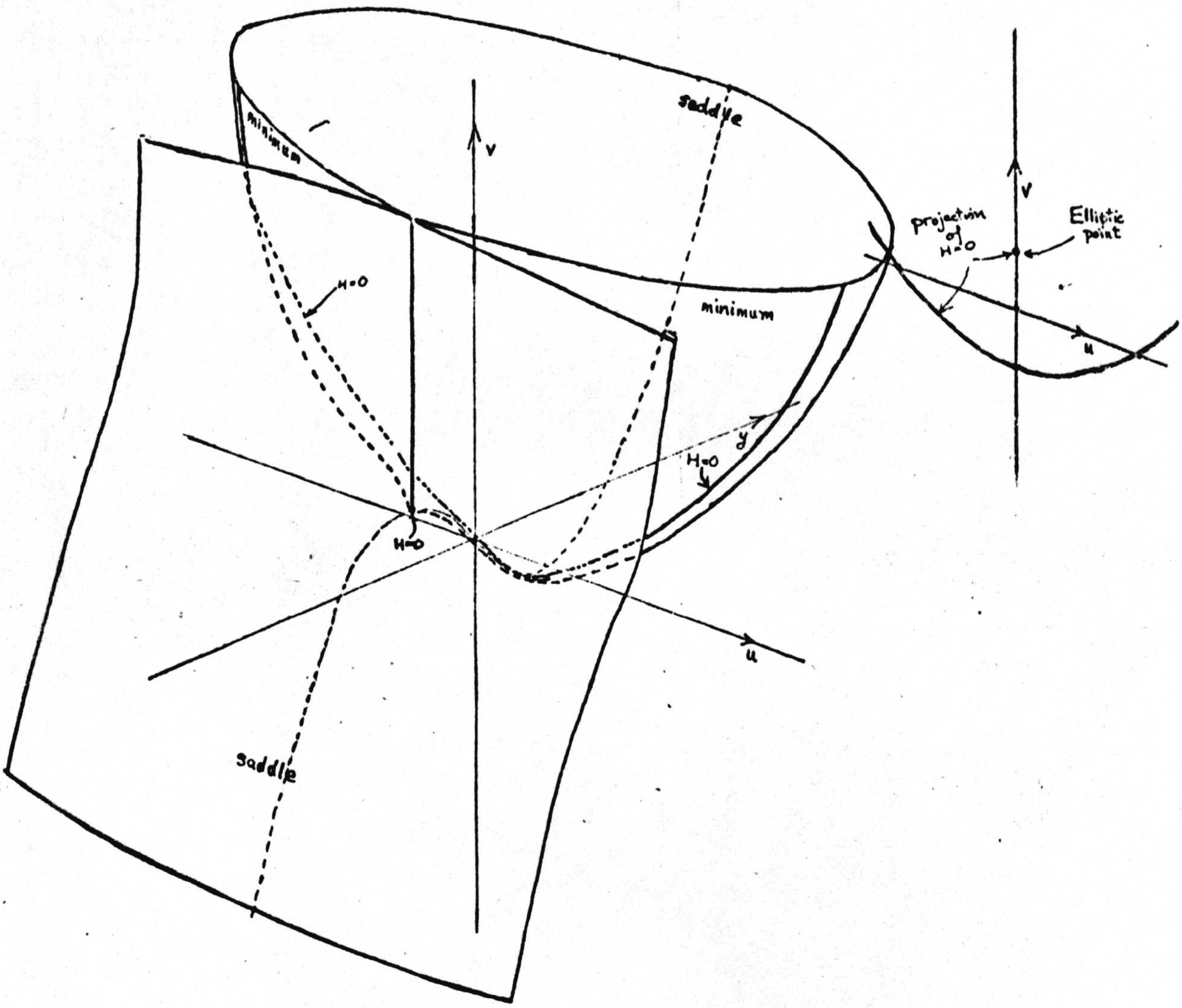
Fig 3


H=O
minimum
saddle
V
minimum
H=O
y
H=O
maximum
u
saddle


V
projection of H=O
u

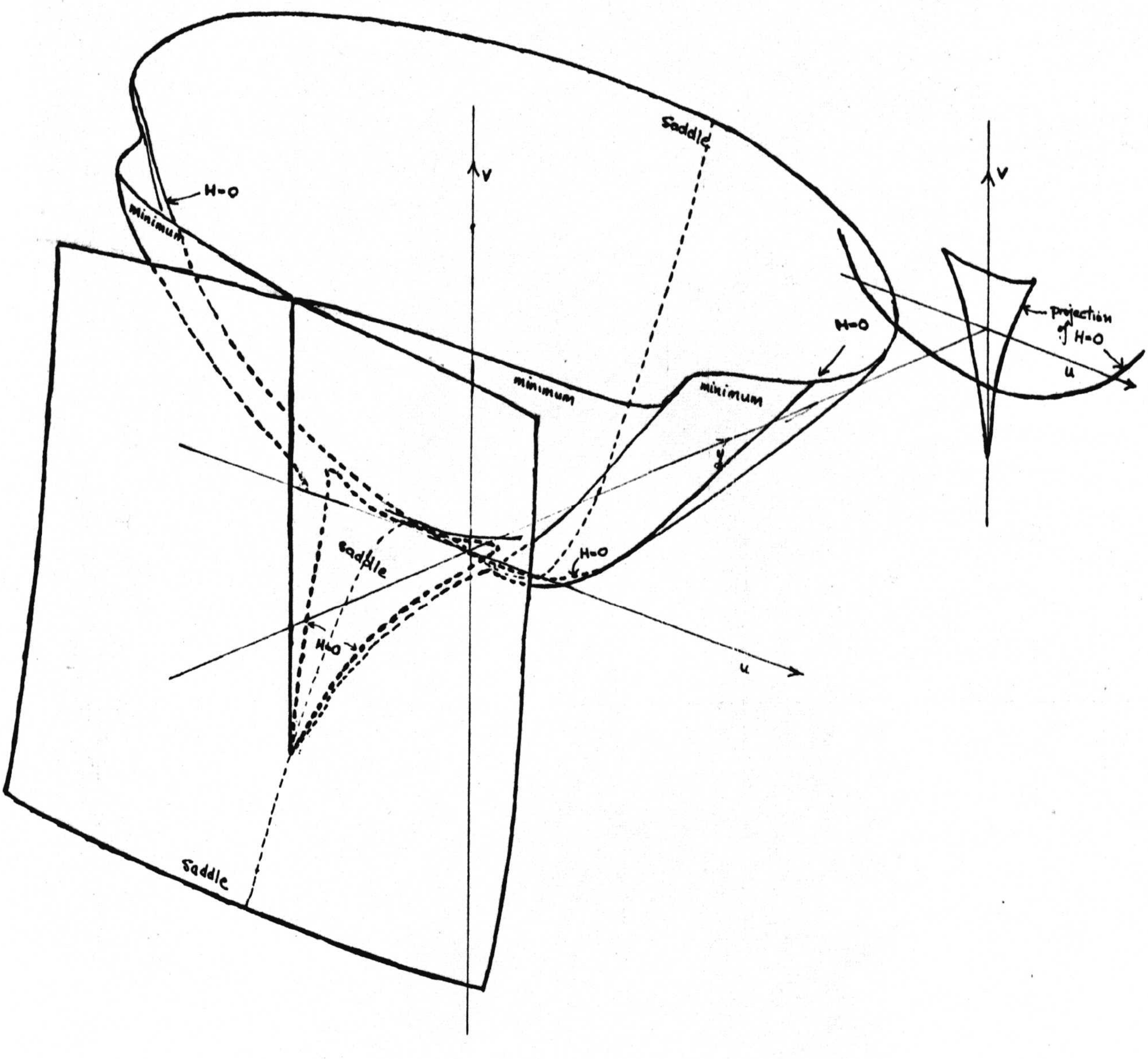
T<0   W=0

Fig 4.



$$\boxed{T<0 \quad w = +\sqrt{-T/6}}$$

Fig 5.



$\boxed{T<0 \quad W>+2\sqrt{-\frac{T}{6}}}$

Fig. 6

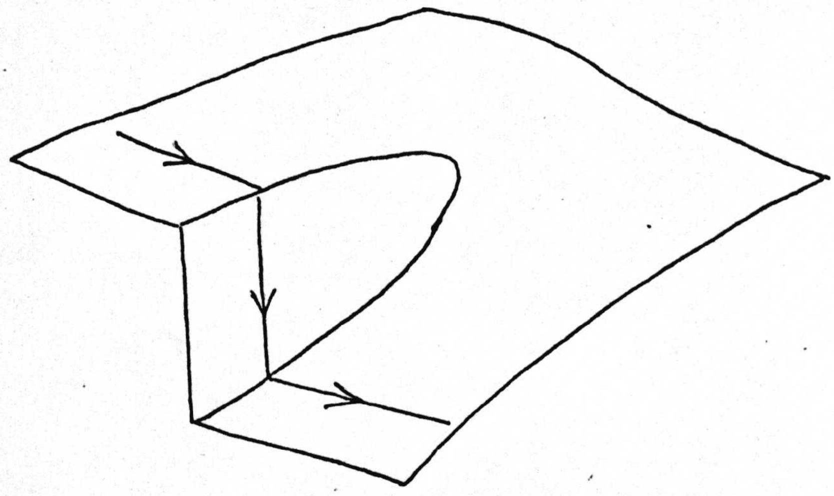


Fig. 7a

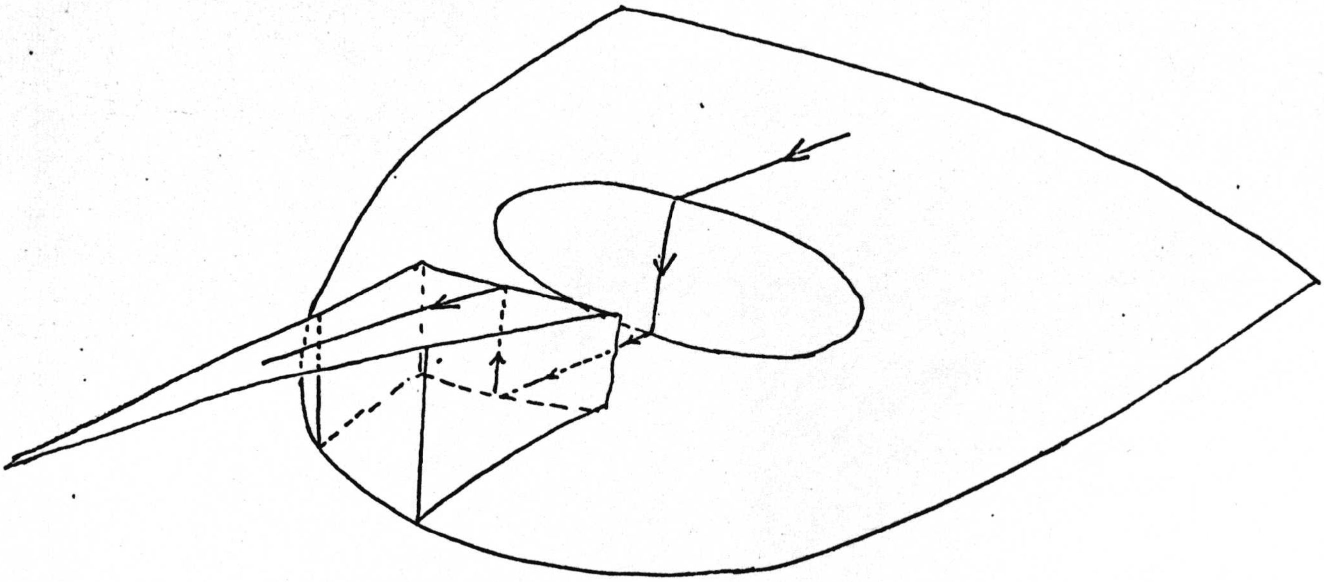


Fig 7b.

Fig 7.

of critical point. An example of the graphs obtained from (3) and the interpretation in terms of critical points is given in Fig. 1. To obtain the complete information about the critical points as far as position in the xy plane and type is concerned, we can use a graph like Fig. 1. Given u,w,t we can draw the graph and for any V a horizontal line will intersect the graph to give y-values of critical points and their type. The x-value corresponding to the points so found can be found from the other graph $v = u/2(y + w)$ as can be seen from equation (2).

For a fixed w,t pair we can vary u to get a three dimensional picture in (u,v,y) - space, which on projection to (u,v) - plane gives as points of B (cf. Appendix I) the projection of stationary points given by $H = 0$ (see (4).) Thus we get a lifting of the two dimensional u,v diagrams of Figs. 5-7, 9-11 of Appendix I, separating the individual critical points. In Figs. 2-6 of this chapter, we give these three dimensional pictures corresponding to $t<0$ for several values of w. These correspond to the inset diagrams of Fig. 5 of Appendix I. The corresponding diagrams for other (u,v) sections can fairly easily be visualised if not very easily drawn and are omitted.

In all the calculations and graphs so far noted in this section, no account has been taken of the value of the potential function. This value will be important when the Maxwell convention is to be used. The information given about the Maxwell set in Appendix I is sufficient to add in the extra surface to Fig. 6. In Fig. 7b we give the y-variable surface corresponding to the diagram of Fig. 7a relating to the R-H catastrophe.

§ 2 The Map $\mathbb{R}^6 \to \mathbb{R}^5$

Morin [2, section V] in a paper relating to Boardmann singularities, Σ^{I}, has given equations for the singularity sets of a map related to the parabolic umbilic. This is the map

$$\phi : \mathbb{R}^6 \longrightarrow \mathbb{R}^5$$

$$(x,y,u,v,w,t) \longmapsto (x^2y + y^4 - ux - vy + wx^2 + ty^2, u,v,w,t)$$

which as a germ is the stable unfolding of the unstable germ given by the V_p defined at the beginning of §1 of this chapter. The method of calculation of unfoldings and this result are both given by Wall [3, Lecture 2, pp.4-5].

The equations Morin gives are for subsets of $\mathbb{R}^6$, but projecting to the (u,v,w,t) coordinates we get curves and surfaces relating to the

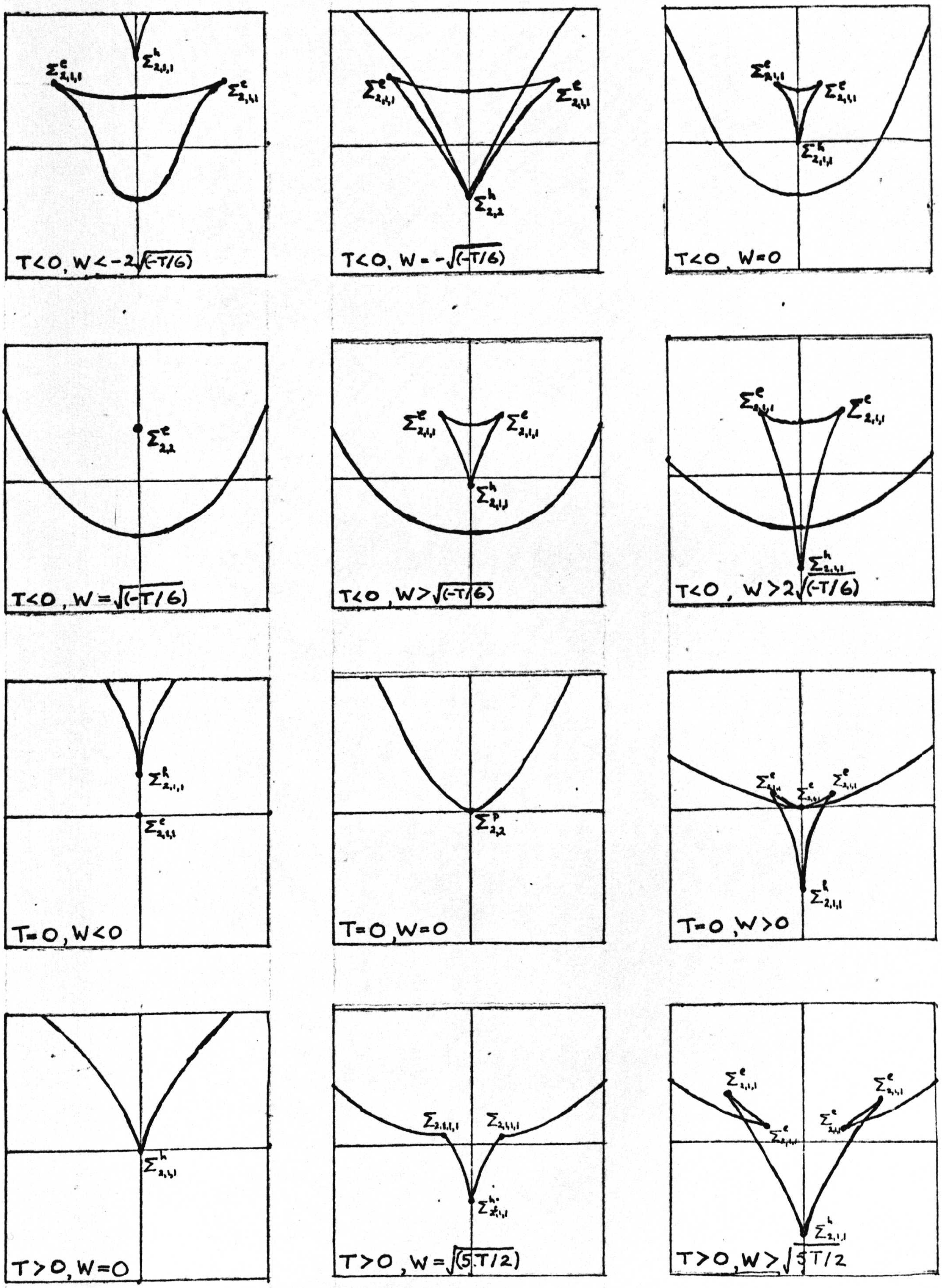

$\Sigma^e_{2,1,1}$
$\Sigma^h_{2,1,1}$
$\Sigma^e_{2,1,1}$
T<0, W<-2√(-T/6)
$\Sigma^e_{2,1,1}$
$\Sigma^e_{2,1,1}$
$\Sigma^h_{2,2}$
T<0, W=-√(-T/6)
$\Sigma^e_{2,1,1}$
$\Sigma^e_{2,1,1}$
$\Sigma^h_{2,1,1}$
T<0, W=0
$\Sigma^e_{2,2}$
T<0, W=√(-T/6)
$\Sigma^e_{2,1,1}$
$\Sigma^e_{2,1,1}$
$\Sigma^h_{2,1,1}$
T<0, W>√(-T/6)
$\Sigma^e_{2,1,1}$
$\Sigma^e_{2,1,1}$
$\Sigma^h_{2,1,1}$
T<0, W>2√(-T/6)
$\Sigma^h_{2,1,1}$
$\Sigma^e_{2,1,1}$
T=0, W<0
$\Sigma^p_{2,2}$
T=0, W=0
$\Sigma^e_{2,1,1}$
$\Sigma^e_{2,1,1}$
$\Sigma^e_{2,1,1}$
$\Sigma^h_{2,1,1}$
T=0, W>0
$\Sigma^h_{2,1,1}$
T>0, W=0
$\Sigma_{2,1,1,1}$
$\Sigma_{2,1,1,1}$
$\Sigma^{h*}_{2,1,1}$
T>0, W=√(5T/2)
$\Sigma^e_{2,1,1}$
$\Sigma^e_{2,1,1,1}$
$\Sigma^e_{2,1,1}$
$\Sigma^e_{2,1,1}$
$\Sigma^h_{2,1,1}$
T>0, W>√5T/2

Fig. 8.

pictures of §1 and Appendix I. The set B of the appendix is the closure of $\Sigma^{1}_{2,1}(\emptyset)$ so projected, most points belonging to $\Sigma_{2,1}(\emptyset)$ and the exceptional points belonging to other Σ^{I} sets as indicated in Fig. 8. The surfaces of Fig. 2-6 of §1 of this chapter are sections of $\Sigma^{1}_{2}(\emptyset)$ projected to (y,u,v,w,t) coordinates, the H = 0 curves being $\overline{(\Sigma^{1}_{2,1}(\emptyset))}$. One peculiarity of these diagrams is the appearance of the singular line $u = 0$, $y = -w$, $v > (6y^2 + t)$ which does not appear in equations for any of the singularity sets. However, lifting one higher dimension we see that, the line itself would correspond to $x = 0$ whereas the surface nearby corresponds to $x \to \infty$ as $y \to -w$. The other set of points that appears geometrically to be misclassified is the set $x = y = t = u = v = 0$, $w > 0$, ($T = 0$, $W > 0$ diagram of Fig. 8), the "beak to beak" points. According to Morin's equations they are ordinary $\Sigma^{e}_{2,1,1}$ points, but from the pictures of Appendix I they seem to be a special case. This special nature is not destroyed by lifting to higher dimensions as could be seen by constructing the diagram corresponding to those of §1.

§3 The Map $\mathbb{R}^4 \to \mathbb{R}^4$

There is a relationship between the elliptic and hyperbolic umbilics as given by Thom [4] and treated in terms of §§1,2 and the last two map germs $(\mathbb{R}^4, o) \to (\mathbb{R}^4, o)$ given by Arnold [5, p.16]. We now give a derivation of this connection for the hyperbolic umbilic, the elliptic umbilic derivation being directly analogous.

The hyperbolic umbilic is given as the family of potentials

$$h : \mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow \mathbb{R} \times \mathbb{R}^3$$
$$(\,(x,y),(u,v,w,)\,) \longmapsto (x^3 + y^3 - ux - vy + wxy,\ (U,V,W)\,)$$

and we can construct in $\mathbb{R}^5$ the set $\Sigma(h)$, the points (x,y,u,v,w) such that the map $(x,y) \longmapsto (x^3 + y^3 - ux - vy + wxy)$ for a given u,v,w has a critical point at (x,y). This set is given by the equations

$$3y^2 - v + wx = 3x^2 - u + wy = 0$$

and from this we derive the mapping $g_h : \mathbb{R}^3 \to \mathbb{R}^3$ given by,

$$U = 3x^2 + wy \;;$$
$$V = 3y^2 + wx \;;$$
$$W = w \,.$$

This map g_h is related to the applications of the hyperbolic umbilic in that the image of the singular set $\Sigma^1(g_h)$ is just the bifurcation set used in the applications. To see that this map does not have a stable germ at the origin we begin with the restriction of g_h

$$g_h| : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (U = 3x \ , \ V = 3y \ )$$

and use Wall's unfolding method [3, Lecture 2, p.4] to get the stable germ at the origin of the map $\hat{g}_h : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$U = 3x^2 + wy \ ;$$
$$V = 3y^2 + tx \ ;$$
$$W = w \ ;$$
$$T = t \ .$$

The map $\hat{g}_h$ includes as an unstable restriction the map g_h. The map $\hat{g}_h$ is not, however, precisely the same as any of the maps given by Arnold [5, p.16] in his complete list of stable map germs. Since $\hat{g}_h$ is stable, it must be equivalent to one of them and we can prove that it is equivalent to f+ given by Arnold i.e.

$$y_1 = x_1 \ ;$$
$$y_2 = x_2 \ ;$$
$$y_3 = x_3^2 + x_4^2 + x_1 x_3 + x_2 x_3 \ ;$$
$$y_4 = x_3 x_4 \ .$$

<u>Lemma II.1</u> $\hat{g}_h$ and f_+ are differentiably equivalent.

<u>Proof A</u> Mather has proved that if two germs are stable then they are differentiably equivalent when the local rings are isomorphic [5, p.19]. Thus since $\hat{g}_h$ and f_+ are both stable we can prove equivalence by finding their local rings and showing they are isomorphic.

The local ring for f_+ is given by Arnold [5, p.18] as a ring of polynomials over $\mathbb{R}$ in two indeterminates with elements of the form $a + bx_3 + cx_4 + dx_4^2$ and multiplication table

	1	x_3	x_4	x_4^2
1	1	x_3	x_4	x_4^2
x_3	x_3	$-x_4^2$	0	0
x_4	x	0	x_4^2	0
x_4^2	x_4^2	0	0	0

The local ring for $\hat{g}_h$ can be calculated as a ring of polynomials over in two indeterminates with elements of the form $a + bx + cy + dxy$ and a multiplication table

	1	x	y	xy
1	1	x	y	xy
x	x	0	xy	0
y	y	xy	0	0
xy	xy	0	0	0

The isomorphism of these rings is then defined by

$$x_3 \longrightarrow \tfrac{1}{\sqrt{2}}(x - y) ;$$

$$x_4 \longrightarrow \tfrac{1}{\sqrt{2}}(x + y) .$$

Hence the two germs are differentiably equivalent. □

<u>Remark</u> The linear differential equivalence of the maps

$$g_h\big| : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (U = 3x^2,\ V = 3y^2)$$

$$f_+\big| : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x_3, x_4) \longmapsto (y_3 = x_3^2 + x_4^2,\ y_4 = x_3 x_4).$$

defined by $x_3 = \sqrt{\tfrac{3}{2}}\,(x - y) \qquad y_3 = (U + V)$

$\qquad\qquad\ \ x_4 = \sqrt{\tfrac{3}{2}}\,(x - y) \qquad y_4 = \tfrac{1}{2}(U - V)$

does not extend to a linear equivalence of $\hat{g}_h + f_+$. It is necessary to reduce to the algebraic invariants to make this equivalence work.

<u>Proof B</u> Another condition for differentiable equivalence is given by Arnold [5, p.14] relating to the quadratic differential of the germs under consideration.

First we construct the quadratic differential at the point in question, so that by Arnold's method in terms of coordinates [5, p.13] we get for at the origin

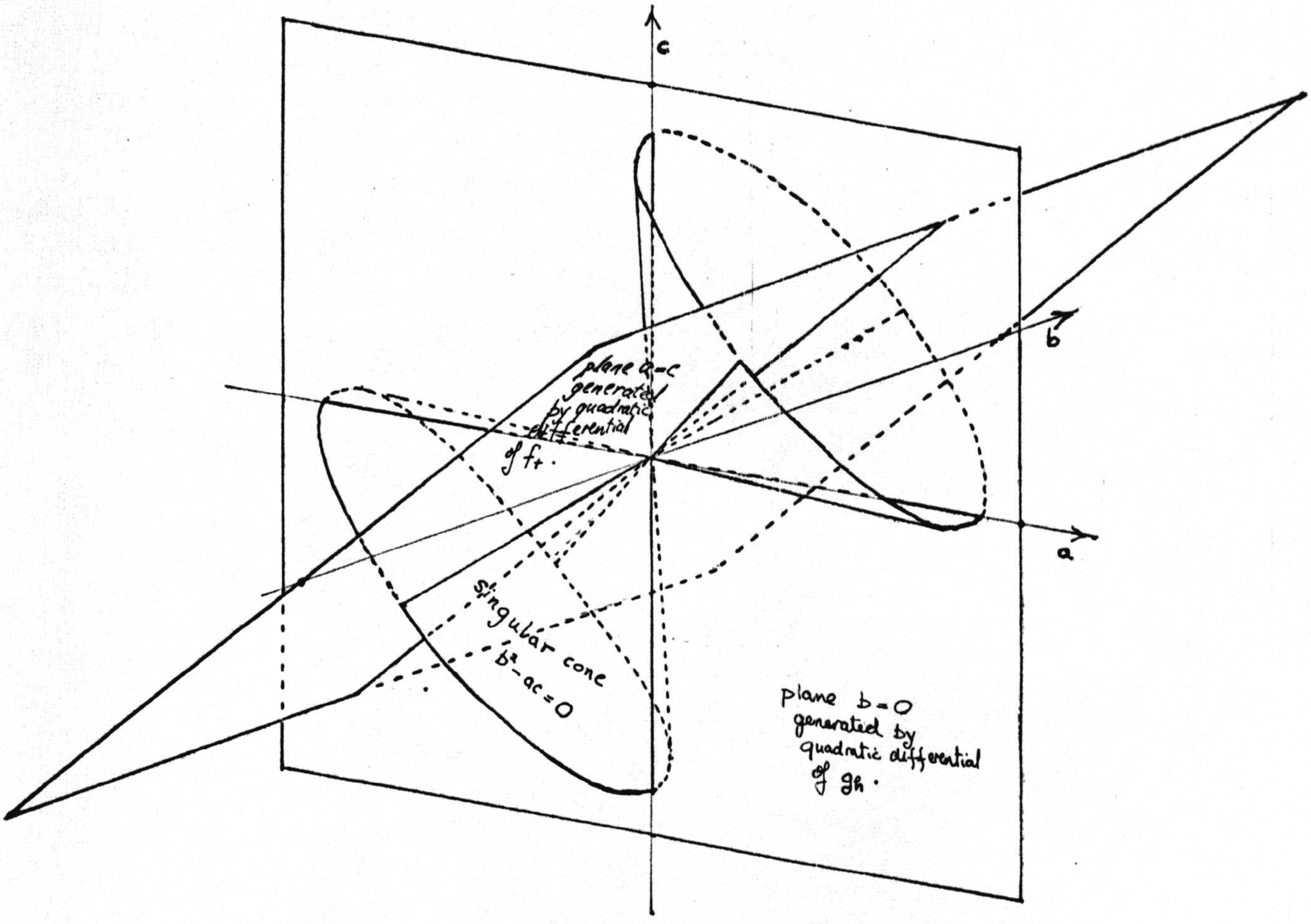
$$Q_{f_+} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 :$$
$$(\xi_1, \xi_2) \mapsto (\xi_1^2 + \xi_2^2,\ \xi_1\xi_2)$$

Then we construct the plane in the (3-dim. linear) space $F(\mathbb{R}^2)$, of quadratic forms $a\xi_1^2 + b\xi_1\xi_2 + c\xi_2^2$ in two variables, generated linearly by $\xi_1^2 + \xi_2^2$ and $\xi_1\xi_2$. The disposition of this plane w.r.t. the cone of singular forms in $F(\mathbb{R}^2)$ is then a differential invariant. For f_+ we get the plane $a = c$ intersecting the cone, see Fig. 9. Using the same calculation for $\hat{g}_h$ we get at the origin

$$Q_{\hat{g}_h} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(\xi_1, \xi_2) \mapsto (6\xi_1^2,\ 6\xi_2^2)$$

and this gives rise to the plane $b = 0 \subset F(\mathbb{R}^2)$ which again intersects the

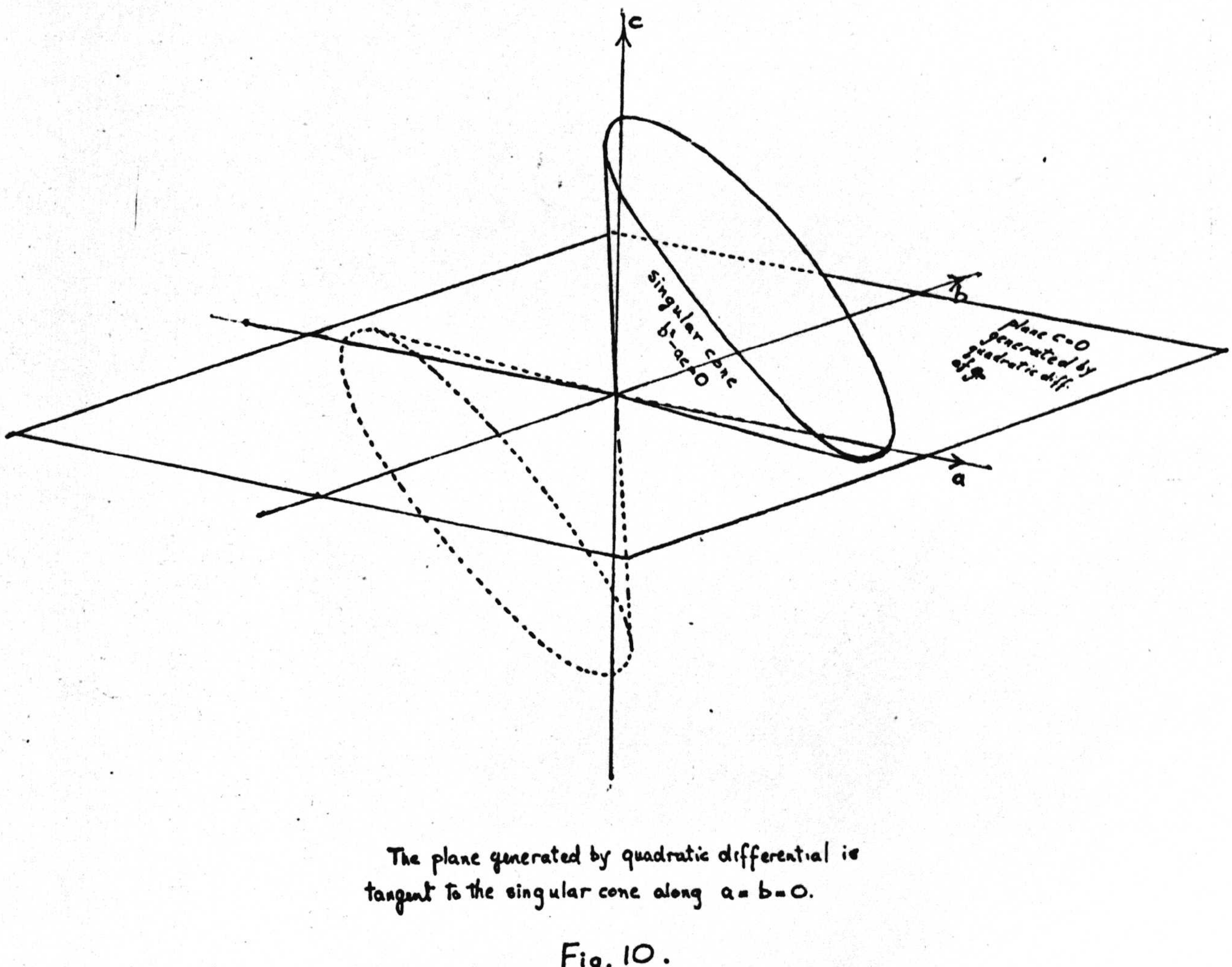
The Hyperbolic Umbilic Map Quadratic Diff.



The diagram shows both planes intersecting the cone of singular quadratic forms in a pair of lines.

Fig. 9.

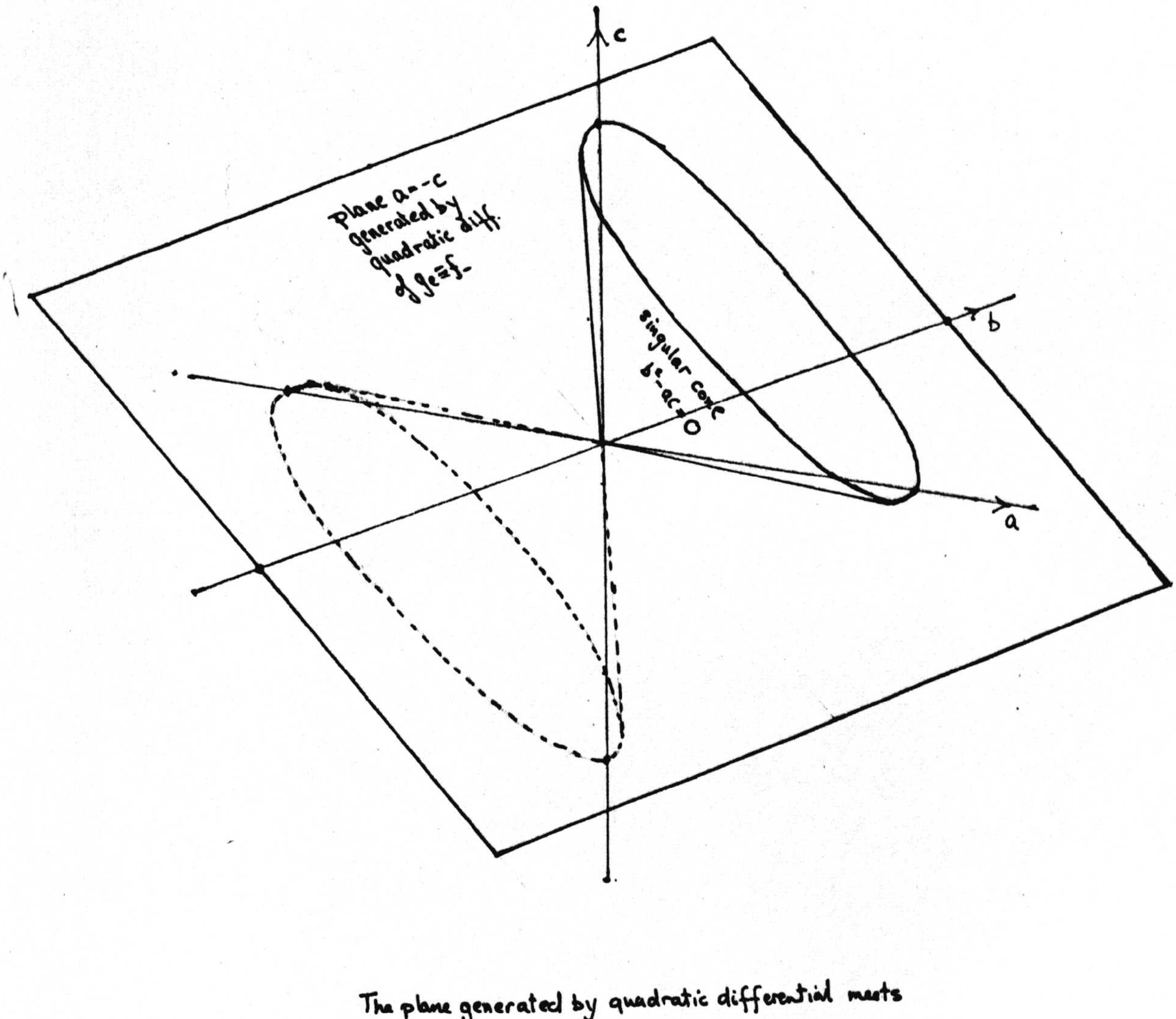
The Parabolic Umbilic Map Quadratic Diff.



The plane generated by quadratic differential is tangent to the singular cone along $a=b=0$.

Fig. 10.

The Elliptic Umbilic Map Quadratic Diff.



The plane generated by quadratic differential meets the cone only at the origin

Fig. 11.

cone and gives Arnold's type 2 [5, p.14], see Fig. 9.

Hence f_+ , $\hat{g}_h$ are in the same orbit of $H(2,2)$ under the action of $GL(2,\mathbb{R}) \times GL(2,\mathbb{R})$ [5. p.14] and hence differentiably equivalent. □

If we now begin with the parabolic umbilic

$$p : \mathbb{R}^2 \times \mathbb{R}^4 \longrightarrow \mathbb{R} \times \mathbb{R}^4$$

$$((x,y),(u,v,w,t)) \mapsto (x^2y + y^4 - ux - vy + wx^2 + ty^2,(U,V,W,T))$$

and work through the process used on the hyperbolic umbilic we get first the map g_p given by

$$U = 2xy + 2wx$$
$$V = x^2 + 4y^3 + 2ty$$
$$W = w$$
$$T = t$$

corresponding to g_h derived above. The set $g_p(\Sigma^1(g_p))$ is just the bifurcation set B given in Appendix I. To show that g_p does not have a stable germ at the origin we start from

$$g_p| : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \mapsto (U = 2xy,\ V = x^2 + 4y^3)$$

and use the method of unfolding referred to above to get the map $\hat{g}_p : \mathbb{R}^5 \to \mathbb{R}^5$

$$U = 2xy + 2wx$$
$$V = x^2 + 4y^3 + 2ty + 2sx$$
$$W = w$$
$$T = t$$
$$S = s$$

which has a stable germ at the origin. This contains g_p as an unstable restriction.

Continuing the analysis of the germ at the origin of the map g_p we look at the local ring and find that it is represented by a ring of polynomials over the reals with elements of the form $a + bx + cy + dx^2 + ey^2$ with multiplication table

	1	x	y	x^2	y^2
1	1	x	y	x^2	y^2
x	x	x	0	0	0
y	y	0	y^2	0	$-x^2$
x^2	x^2	0	0	0	0
y^2	y^2	0	$-x^2$	0	0

This ring is also the local ring for the germ of the map $g_p|$ and we have the three umbilics related to maps $\mathbb{R}^2 \to \mathbb{R}^2$ each with an isolated Σ^2

point at the origin. The local rings of map germs with singularity type Σ^2 have been classified by Mather [5, p.19] and we have the following table

	map $\mathbb{R}^2 \to \mathbb{R}^2$	Ring type	Codimension	
hyperbolic umbilic	$(x,y) \mapsto (xy, x^2 + y^2)$	$I_{2,2}$	2	Table 1
elliptic umbilic	$(x,y) \mapsto (xy, x^2 - y^2)$	$II_{2,2}$	2	
parabolic umbilic	$(x,y) \mapsto (xy, x^2 + y^3)$	$I_{2,3}$	3	

Further Arnold gives unfoldings for all $I_{m,n}$, $II_{m,n}$ types [5, p.19] and for $I_{2,2}$ and $II_{2,2}$ these turn out to be just the maps f_+, f_-. The standard unfolding for g_p however is given as

$$
\begin{aligned}
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= x_3 \\
y_4 &= x_4 \; x_5 \\
y_5 &= x_4^2 + x_5^3 + x_1 x_4 + x_2 x_5 + x_3 x_5^2
\end{aligned}
$$

rather than $\hat{g}_p$, but the calculation for $\hat{g}_p$ was not uniquely determined and we could easily have arrived at the standard answer.

Relating to Proof B of Lemma II.1 we can calculate the quadratic differential of g_p at the origin and get

$$
\begin{aligned}
Q_{g_p} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\
(\xi_1, \xi_2) &\longmapsto (4\xi_1\xi_2 ,\; 2\xi_1^2)
\end{aligned}
$$

and we generate the plane $c = 0$ in $F(\mathbb{R}^2)$ which touches the singular cone along the axis, $b = c = 0$, see Fig. 10. Previously we showed that the hyperbolic umbilic gave rise to a plane situated like $b = 0$ and we can show that the elliptic umbilic gives the plane $a = -c$. In Figs. 9-11 we show these planes in $F(\mathbb{R}^2)$ and it is then clear that the parabolic umbilic map is a transitional case between elliptic and hyperbolic when looked at from this point of view.

Looking at the mappings of Table I geometrically, we can see another way in which the parabolic umbilic map is a transition from elliptic to hyperbolic. We draw in Figs. 12-15 sections of the graphs of these maps which show geometrically how the plane is folded in each case. We also show how the singularity sets Σ^1, Σ^2 are mapped by each of the mappings. The $u = 0$ slice through the three-dimensional pictures of Figs. 12-15 shows the differences : Elliptic - maximum at $y = 0$; hyperbolic - minimum at $y = 0$; parabolic - inflexional stationary point at $y = 0$. The similarities are the y = const. sections; for all three maps these are parabolas, except for the half axis case at $y = 0$. Again in all three

Hyperbolic Umbilic Map - Partial Graph of Restriction.

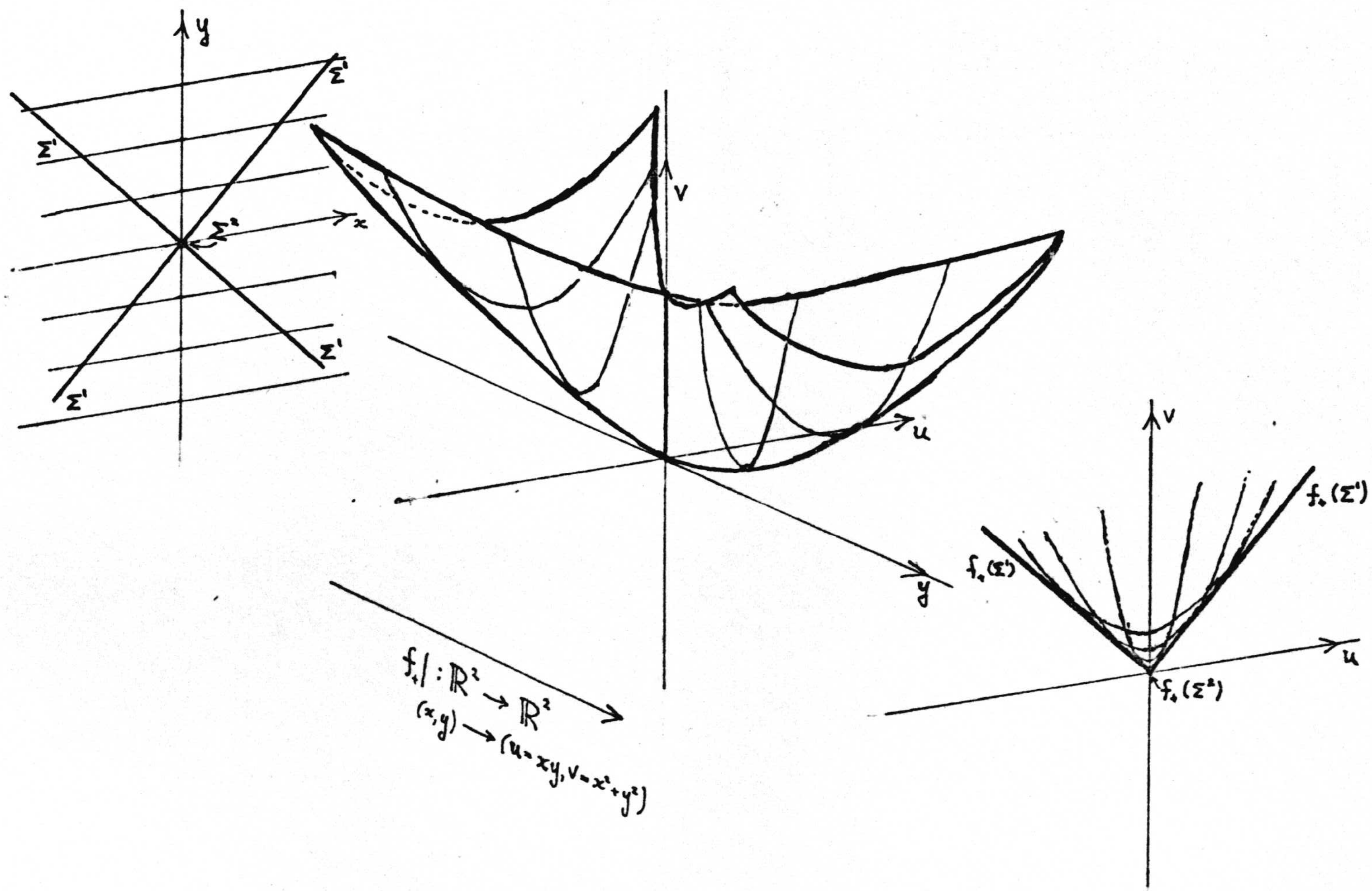


Fig. 12.

Parabolic Umbilic Map - Partial Graph of Restriction.

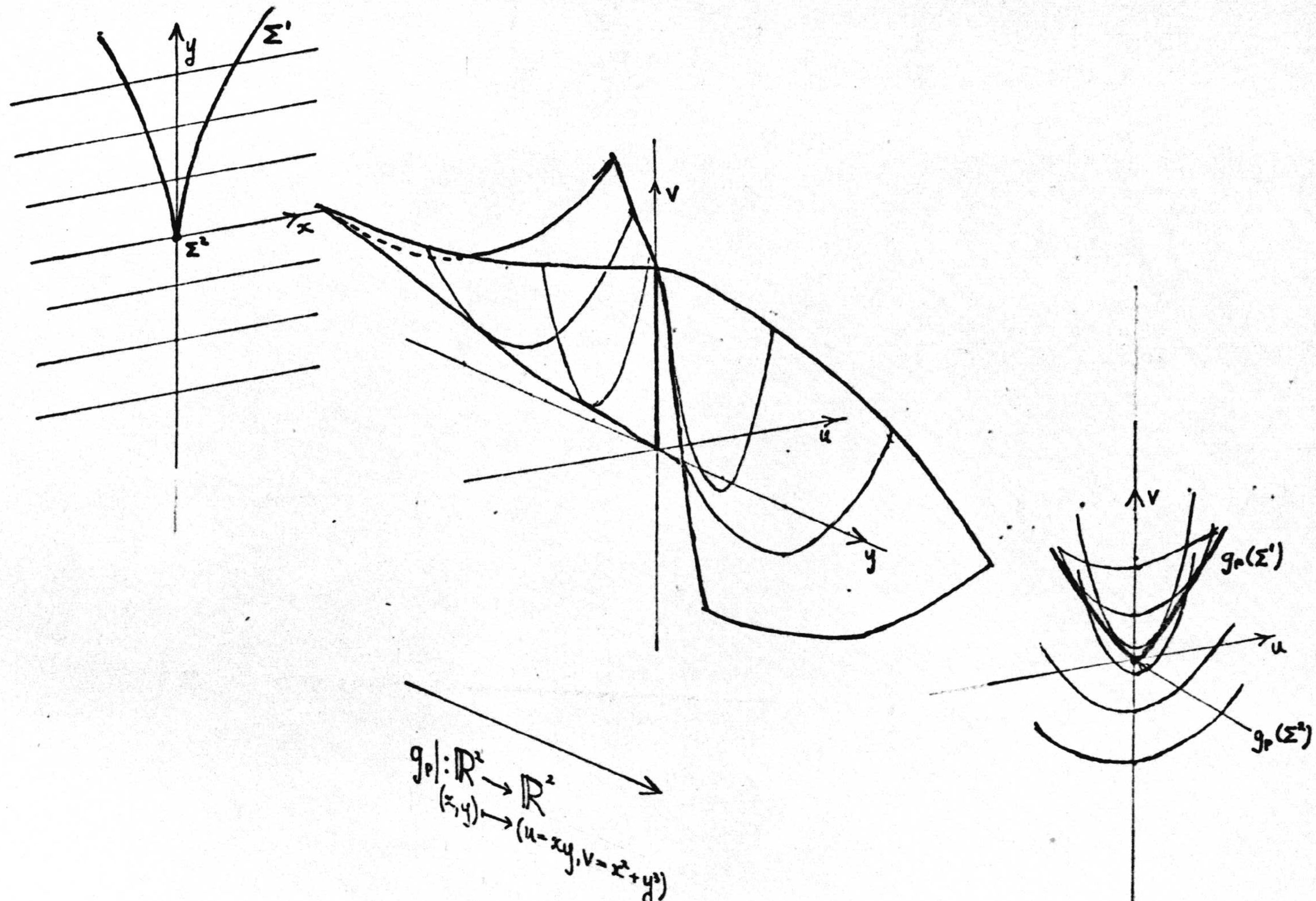


Fig. 13.

Elliptic Umbilic Map - Partial Graph of Restriction

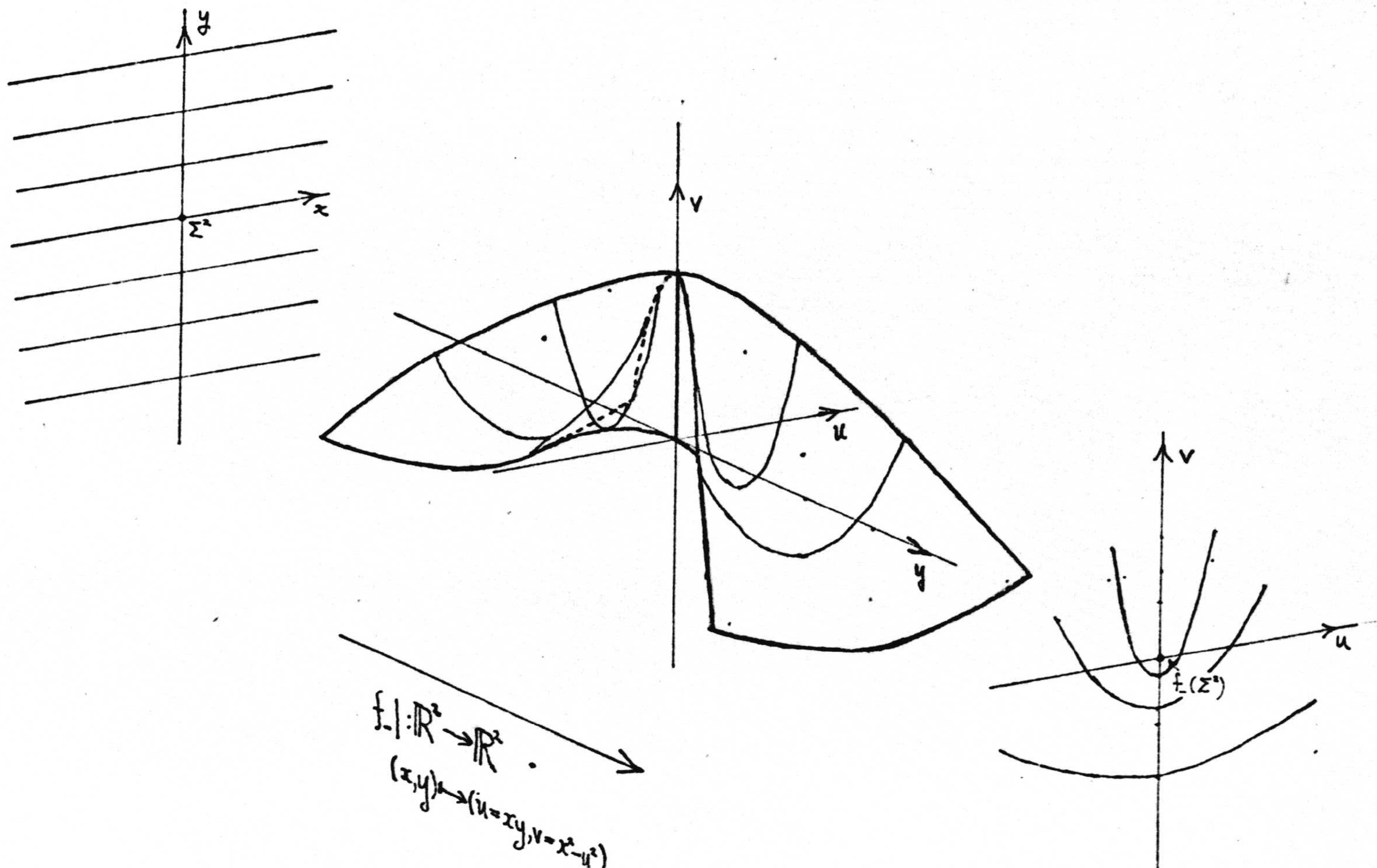


Fig. 14.

maps the $y > 0$ lines given a direction by x-increasing, 'keep the direction' i.e. go to parabolas with u-increasing and the $y < 0$ lines 'switch over directions'.

We have already noted that $g_p(\Sigma^1)$ is the same geometrically as B the bifurcation set of Appendix I and applying Morin's method [2, § V] we can calculate the equations for the other Σ^I. The images of these under g_p can be represented as subsets of B and the two dimensional sections of this representation are given in Fig. 15. Further, the diagrams of §1 (Figs. 2-6) are the analogues of the diagrams of Figs. 12-15 if we consider g_p in terms of a family of maps $\mathbb{R}^2 \to \mathbb{R}^2$. Thus it is not difficult to draw the two dimensional sections ((x,y) - plane) of the singularity sets themselves and relate them to the geometric folding of the plane (cf. illustrations in Figs. 12-15). The singularity sets in (x,y) planes corresponding to the (u,v) diagrams of Fig. 15 are given (in corresponding position) in Fig. 16. To calculate the singularity sets we begin with the Jacobian matrix, J, of g_p and consider its rank.

$$J = \begin{pmatrix} 2(y+x) & 2x & 2x & 0 \\ 2x & 12y^2 + 2t & 0 & 2y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus we get Σ^1 from det (J) = 0 i.e. from equation

$$\delta_1(x,y) \equiv x^2 - (y+w)(6y^2+t) = 0$$

as expected. To find Σ^2 we need values of x, y, w, t which reduce the rank of J to 2. This gives the conditions

$$\delta_2(x,y) \equiv x = 0$$
$$\delta_3(x,y) \equiv y + w = 0$$
$$\delta_4(x,y) \equiv 6y^2 + t = 0$$

i.e. we get $x = 0,\ y = -w = \pm\sqrt{-t/6}$ as equations for Σ^2. So the origin is not an isolated point of Σ^2 and we must consider later subsets $\Sigma^{2,h}$. Since the rank of J is ≥ 2 for all points we see that $\Sigma^3, \Sigma^4, ..$ etc. are empty. We now calculate $\Sigma^{1,1}$ and we need to consider the simultaneous vanishing of determinants given by Jacobians for a selection of 4 functions from U, V, W, T, δ_1. These determinants reduce to the equations,

$$\delta_1(x,y) \equiv x^2 - (y+w)(6y^2+t) = 0 \quad ;$$
$$\delta^*_{1,1}(x,y) \equiv x^2 + \frac{1}{2}(y+w)(18y^2 + 12wy + t) = 0 \quad ;$$
$$\delta^*_{1,2}(x,y) \equiv x\,(2\,(6y^2+t) + (18y^2 + 12wy + t)\,) = 0 \ .$$

which give rise to the algebraic varieties defined by

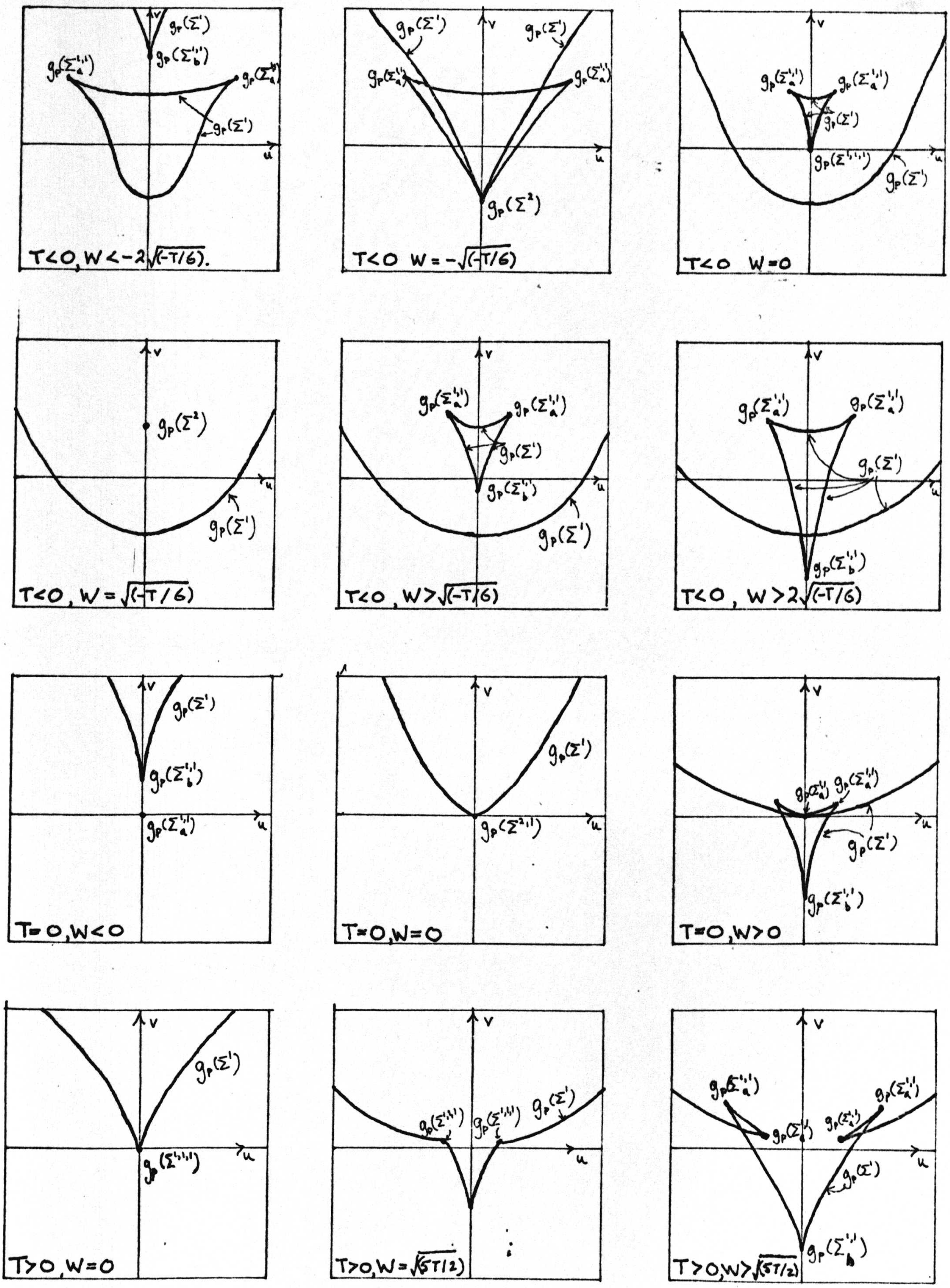
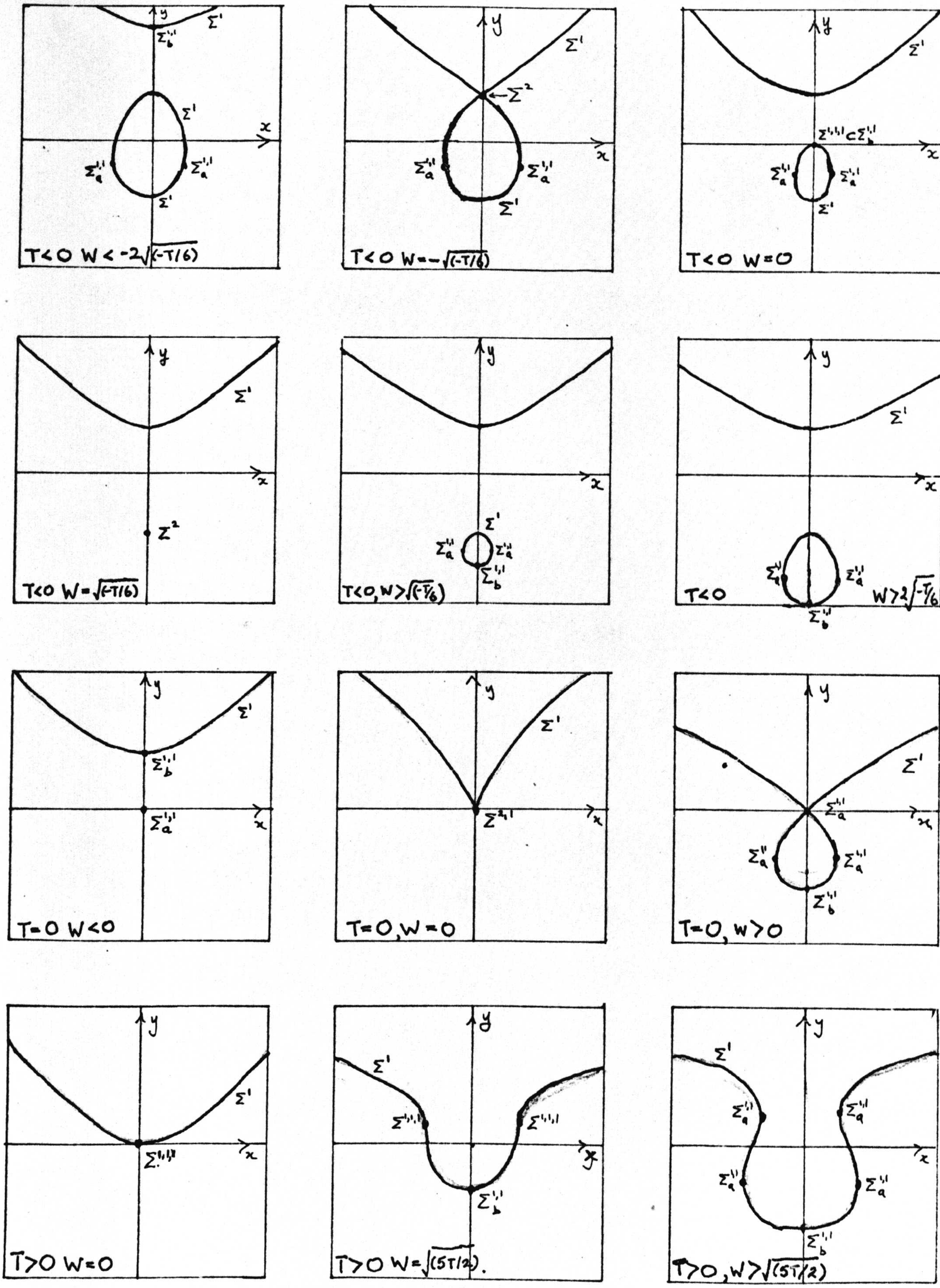

T<0, W<-2√(-T/6).
T<0 W=-√(-T/6)
T<0 W=0
T<0, W=√(-T/6)
T<0, W>√(-T/6)
T<0, W>2√(-T/6)
T=0, W<0
T=0, W=0
T=0, W>0
T>0, W=0
T>0, W=√(5T/2)
T>0, W>√(5T/2)

Fig. 15.



T<0 W<-2√(-T/6)
T<0 W=-√(-T/6)
T<0 W=0
T<0 W=√(-T/6)
T<0, W>√(-T/6)
T<0 W>2√(-T/6)
T=0 W<0
T=0, W=0
T=0, W>0
T>0 W=0
T>0 W=√(5T/2).
T>0, W>√(5T/2)

Fig. 16.

(a) $10y^2 + 4wy + t = 0$ and $x^2 = (y + w)(6y^2 + t)$;

(b) $x = 0$ and (i) $y = -w$;

or (ii) $y = -w = \pm\sqrt{-t/6}$;

or (iii) $y = -w = -w/3 \pm \frac{1}{3}\sqrt{w^2 - t/2}$;

or (iv) $y = \pm\sqrt{-t/6} = -w/3 \pm \frac{1}{3}\sqrt{w^2 - t/2}$.

The four cases of (b) arise from the different ways of solving the equations remaining after putting $x = 0$. Clearly (i) includes (ii) and (iii) so that they give no extra points. Looking at (iv) we see that this is equivalent to a condition for the quadratics $6y^2 + t$, $18y^2 + 12wy + t$ to have a common root. The resultant determinant of these two quadratics is just $144t(6w^2 + t)$ and the vanishing of this can be shown to be included in the $x = 0$, $y = -w$ case. Thus $\Sigma^{1,1}$ is defined as the subsets of Σ^1 given by

$$\Sigma_a^{1,1} = \{(x,y,w,t) \mid \in \Sigma^1 \text{ and } 10y^2 + 4wy + t = 0\} ;$$

$$\Sigma_b^{1,1} = \{(x,y,w,t) \mid x = 0 \text{ and } y = -w\}.$$

For $\Sigma^{1,1,1}$ following Morin's calculations, we use only the 4 x 4 determinants having 2 non-trivial rows and get as an extra condition, above those given for $\Sigma^{1,1}$, only $y = -w/5$. This defines a subset of $\Sigma_a^{1,1}$ and the subset defined by $w = 0$ in $\Sigma_b^{1,1}$. Similarly using 3 x 3 determinants we get only the origin, $x = y = w = t = 0$ in the set $\overline{\Sigma^{2,1}}$ so that since $\Sigma^{2,2}$ is empty we have that $\Sigma^{2,1}$ is the origin.

§4 Conclusions and Possible Extensions

The chapter has considered some aspects of the geometry of the parabolic umbilic but has exhausted neither the possible lines of investigation nor each topic in itself. In this section we look at possible extensions.

Relating to §1 and Appendix I it would be possible to consider in more detail the geometry of the relationship of the state representation by a point in a plane and the control variables u,v,w,t (cf Zeeman, [1]). From here it would be possible to construct applications of this catastrophe comparable with those given by Thom and Zeeman for other elementary catastrophes. The figures of §1 and Appendix I indicate fairly adequately how to visualise all parts of such a model.

In §2 we have just noted a connection with an apparently divergent topic of investigation. The further developments would not appear to be too closely related to the work of this chapter.

The section of this chapter which suggests most further investigation is §3. Directly related to the parabolic umbilic is the question of why we get an extra 'degree of freedom' in the unfolding of the map $\mathbb{R}^2 \to \mathbb{R}^2$ as compared with the family of potential functions. In the cases of the hyperbolic and elliptic umbilic Zeeman has given a rough explanation [6] relating to looking at the regions of the (x,y) planes cut out by Σ^1 (cf. Fig. 16) in terms of max, min and saddle. We can make this a little more precise by looking at the sign of the determinant which is the Jacobian for the map $\mathbb{R}^2 \to \mathbb{R}^2$ and the Hessian of the potential function but even then it does not fully explain the parabolic umbilic. In this case we need besides Σ^1 an extra dividing line corresponding to a critical point of the potential function being at infinity. A full explanation of this and the general relationship between the unfoldings of $f: \mathbb{R}^n \to \mathbb{R}$ and $\underline{\nabla} f: \mathbb{R}^n \to \mathbb{R}^n$ appears to be of interest.

Relating to this would be an investigation of the geometry of the map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ given in Mather's list [5, p.19] from the point of view of Figs. 12-16. This would give a picture of the Σ^2 singularity types and their unfoldings in straightforward visual terms.

In a more general way it could well be of interest to look at aspects such as - the relationship of the unfoldings to the homology (as defined by Morse) of the critical points; the calculation of the homotopy groups of $\mathbb{R}^4 - B$; the precise connection with Porteous's normal singularities [7].

It is thus evident that although we have a fair amount of geometric information on the parabolic umbilic, there is still more of interest to be calculated.

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CHAPTER III

A SECTION OF THE DOUBLE RIEMANN-HUGONIOT CATASTROPHE

Some of the general properties of the double R-H catastrophe, i.e. the unfolding of

$$V_{db} : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x,y) \longmapsto (x^4 + y^4)$$

are described in Chapter IV. In this chapter we describe details of one four-dimensional section of the unfolding which is given by Thom [1] as a version of the parabolic umbilic. The fact that this version of the parabolic umbilic should rather be considered as a section of the double R-H catastrophe was suggested by Zeeman [2].

§1 The Basic Equations

If we consider the map germ

$$f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$$

$$(x,y) \longmapsto x^2y + \left(\frac{x^4 + y^4}{4}\right)$$

we get as an unfolding

$$\hat{f} : (\mathbb{R}^2 \times \mathbb{R}^4, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^4, 0)$$

$$((x,y),(u,v,w,t)) \longmapsto (x^2y + \frac{x^4+y^4}{4} - ux - vy + wx^2 + ty^2, u,v,w,t) .$$

Consider now the family of potential functions,

$$V_c : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x,y) \longmapsto x^2y + \frac{x^4+y^4}{4} - ux - vy + wx^2 + ty^2 \qquad \text{-------- (1)}$$

and we get as equations for a critical point of (1)

$$\frac{\partial V_c}{\partial x} = \frac{\partial V_c}{\partial y} = 0 ;$$

i.e. $u = x\left[2(y + w) + x^2\right]$ -------------- (2)

$v = x^2 + y^3 + 2ty$ ------------------ (3)

For a bifurcation point we get the condition

$$H(V_c) = \begin{vmatrix} 2(y + w) + 3x^2 & 2x \\ 2x & 3y^2 + 2t \end{vmatrix} = \text{Hessian of } V_c = 0 ,$$

i.e. $x^2 (9y^2 + 6t - 4) + 2(y + w)(3y^2 + 2t) = 0$ ------------- (4)

which for $y \neq \pm\sqrt{\left(\frac{2}{3}\left(\frac{2}{3} - t\right)\right)}$ gives

$$x^2 = -2(y + w)(y^2 + 2t/3) / 3(y^2 + 2t/3 - 4/9) \qquad \text{--------- (5)}$$

Hence with the restriction stated we can use (5) to give a pair of parametric equations for the curve in the u,v plane given by (2) and (3). Thus we get

$$u = \pm \frac{4(y + w)(y^2 + 2t/3 - 2/3)}{3(y^2 + 2t/3 - 4/9)} \sqrt{\frac{-2(y + w)(y^2 + 2t/3)}{3(y^2 + 2t/3 - 4/9)}} \quad - - - (6)$$

$$v = y^3 + 2ty - \frac{2(y + w)\ (y^2 + 2t/3)}{3(y^2 + 2t/3 - 4/9)} \quad - - - - - - - - - - (7)$$

as equations, parameter y, for the u-symmetric u,v,sections of the bifurcation set for constant w,t. The exceptions are the points given by $y = \pm\sqrt{(2/3\ (2/3 - t))}$, which must be discussed separately. The problem posed by (6) and (7) is analogous to that solved in Appendix I. However the equations (6) and (7) are more difficult to deal with than the corresponding equations of the appendix. The main analytic and computational difficulties relate to special condition $y = \pm\sqrt{2/3\ (2/3 - t)}$ and this will be dealt with at the appropriate places in the subsequent sections. In cases where $t > (2/3)$ or w is such that (6) and (7) can be used with a straight forward infinity at $y = \pm\sqrt{\frac{2}{3}\ (\frac{2}{3} - t)}$ then it is possible to use a simple computer program to plot most features of the u,v graphs for a given w,t. (See Appendix II for computer program). The computer program is, however, not much use unless it is known what values of w,t to use and we shall produce these answers analytically. In this chapter it has not been possible to confirm all the geometric features of the computer results, but a substantial amount of the information needed to draw three-dimensional pictures corresponding to results of Appendix I has been obtained.

§ 2 Ranges for Parameter

We can see from (6) that not all values of y will give real values for u (or x). It is not difficult to calculate the ranges of y for which u,v are real. We can thus construct a three dimensional picture involving y,w,t to show these ranges. For use in the calculations and for easy use in subsequent arguments, we have taken t = const sections of the three dimensional picture. The seven different cases for t = const. are given in Figs. 1-2, with some examples of w = const. sections for completeness. In Fig. 1 (relating to $t \not< 0$) we give also more detailed information about u,v values and these can be carried over to the smaller diagrams of Fig. 2.

Turning to the points $y = \pm\sqrt{\frac{2}{3}\ (\frac{2}{3} - t)}$ we see that in most cases these correspond to a part of a w = const line giving a (u,v)- graph

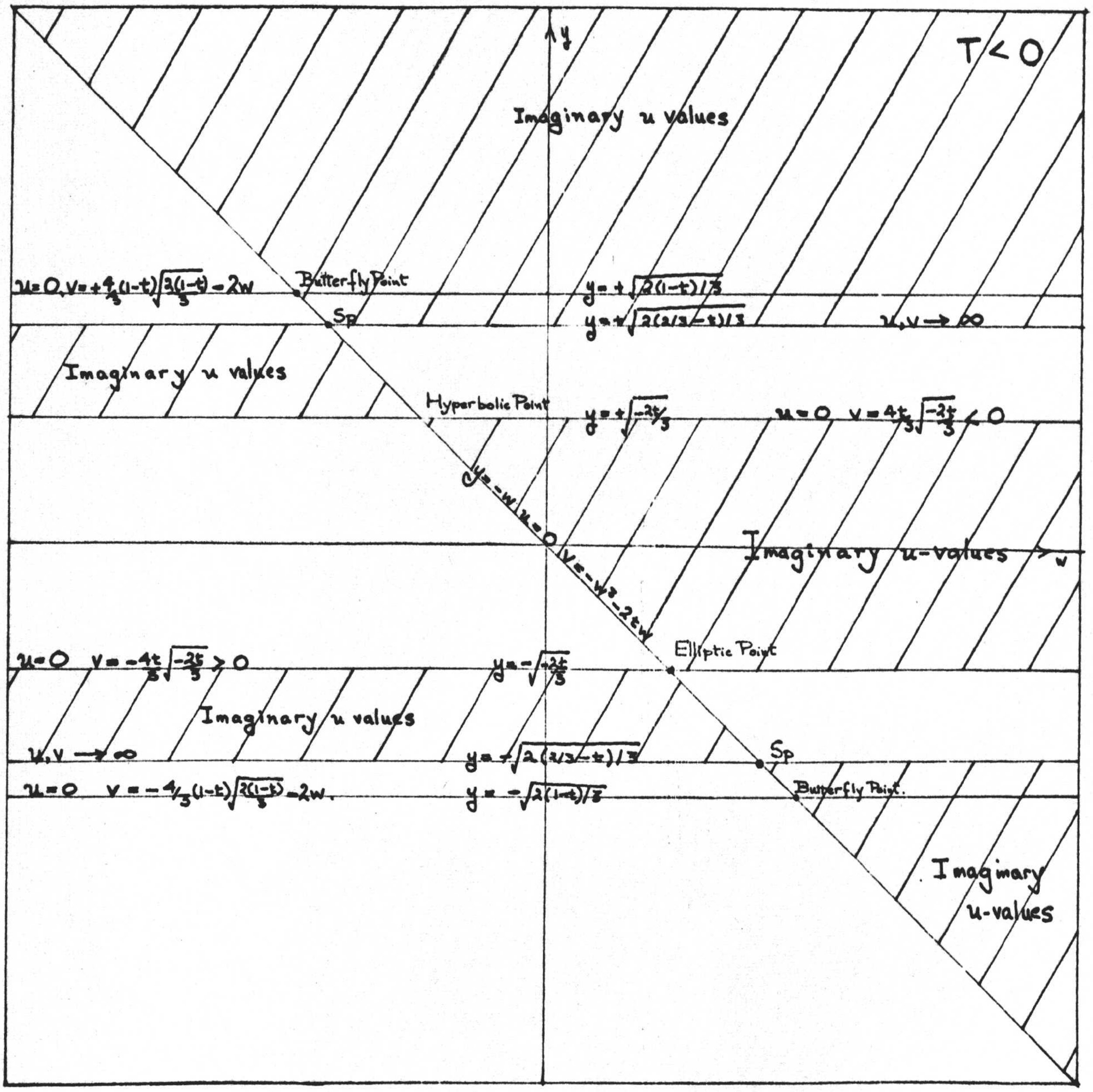

T<0
y
w
Imaginary u values
Butterfly Point
Sp
Hyperbolic Point
Elliptic Point
Imaginary u-values
u, v → ∞
u = 0

Fig 1.

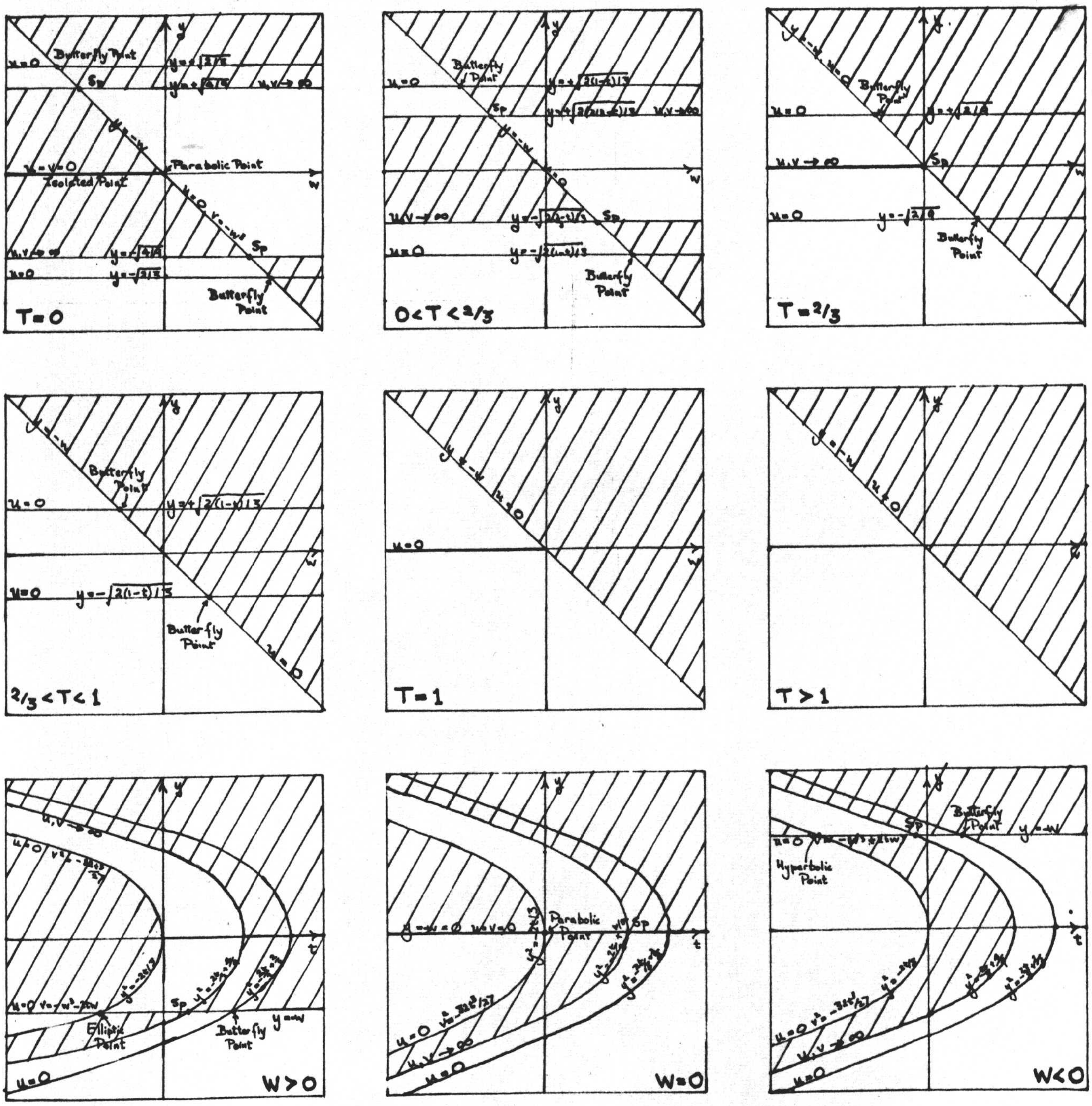


Fig 2.

with a branch having

$$u,v \to \infty \quad \text{as } y \to \pm\sqrt{\tfrac{2}{3}\left(\tfrac{2}{3} - t\right)} .$$

However, for $y = w = \pm\sqrt{\frac{2}{3}(\frac{2}{3} - t)}$ we find (4) gives no condition on x. Thus we have the parametric equations (parameter x)

$$u = x^3 \quad \text{- - - - - - - - - - - - - - - - (8)}$$

$$(v + (w^3 + 2tw)\ ) = x^2 \quad \text{- - - - - - - - - - - - (9)}$$

Thus points marked 'Sp' in Figs. 1-2 correspond to whole branches rather than to single points. The neighbouring y values and the (u,v) values corresponding to $y \to Sp$ will be discussed in section 3.

§3 Detailed Analysis - Intersections with v-Axis

For the curves drawn using parametric equations (6), (7) we can see that the symmetric curves meet $u = 0$ for $y = -w$, $y = \pm\sqrt{\frac{2}{3}(1 - t)}$, $y = \pm\sqrt{-2t/3}$. To analyse the nature of these points we work through the $t < 0$ case as pictured in Fig. 1 and then make the minor alterations for the six simpler $t \geqslant 0$ cases. The results of this section depend on the formulae (6), (7) and the further equations

$$\frac{du}{dy} = -\frac{1}{x}\ \frac{4\ (y + w)}{9\ (y^2 + 2t/3 - 4/9)}\ \left(3y^2 + 2t + 2x\frac{dx}{dy}\right) \quad \text{- - - (10)}$$

$$\frac{dv}{dy} = 3y^2 + 2t + 2x\,\frac{dx}{dy} \quad \text{- - - - - - - - - - - - - - - - (11)}$$

$$\frac{dv}{du} = \frac{-9x\ (y^2 + 2t/3 - 4/9)}{4\ (y + w)} \quad \text{- - - - - - - - - - - - - - (12)}$$

with (10) being restricted by $y \neq \pm\sqrt{\frac{2}{3}(\frac{2}{3} - t)}$, $x \neq 0$ and (12) by the further conditions $3y^2 + 2t + 2x\frac{dx}{dy} \neq 0$, $y \neq -w$. We shall return to the equation $3y^2 + 2t + 2x\frac{dx}{dy} = 0$ in §4 where we consider the cusps. For this section we simply make the following assumption :

<u>Assumption A</u> For any fixed w,t there are only a finite set of y values defined by the equation $3y^2 + 2t + 2x\frac{dx}{dy} = 0$, where $\frac{dx}{dy}$ is found by differentiating (5) with respect to y.

<u>Lemma 1</u> For $t < 0$ $y = -w$, $w < -\sqrt{-\frac{2}{3}(t - 1)}$ we have in the u,v, diagram a cusp on the v-axis, at $v = -(w^3 + 2tw)$ symmetric w.r.t u and pointing in the v-increasing direction.

<u>Proof</u> For $y = -w$, $w < \sqrt{-\frac{2}{3}(t-1)}$, $t<1$ (a fortiori $t<0$) we have $u = 0$, $v = -(w^3 + 2tw)$. Considering Fig. 1 we see that for real points we need $y \leqslant -w$ for such w. For such a neighbourhood of $-w$, $(y + w) \leqslant 0$, $y^2 + \frac{2}{3}(t-1) > 0$, $y^2 + \frac{2}{3}(t - \frac{2}{3}) > 0$. This shows that it is the negative square root for x that corresponds to the positive value of u. Using the negative sign for x in (12) we see that $({}^{dv}/du) < 0$ for some neighbourhood $-w - \epsilon < y < -w$ (using Assumption A) with $u > 0$. Further

$$\left\{\frac{dv}{du}\right\}^2 = \frac{-27}{8}\,\frac{(y^2 + \frac{2}{3}t)\,(y^2 + {}^{2t}/3 - {}^{4}/9)}{(y + w)} \qquad \text{- - - - - - - - - (13)}$$

so that $| dv/du | \to \infty$ as $y \to -w$. Hence we get an upward pointing cusp at $y = -w$. □

<u>Proposition 2</u> For $t < 0$ $w \neq \pm\sqrt{\frac{2}{3}(1-t)}$, $\pm\sqrt{\frac{2}{3}(\frac{2}{3} - t)}$, $\pm\sqrt{-{}^{2t}/_3}$ we have it $y = -w$ a cusp on the v-axis and the direction in which it points is given by the following diagram :

upward		downward		downward		downward		downward		downward		upward
	$-\sqrt{\frac{2}{3}(1-t)}$		$-\sqrt{\frac{2}{3}(\frac{2}{3}-t)}$		$-\sqrt{-\frac{2t}{3}}$	0	$+\sqrt{-\frac{2t}{3}}$		$+\sqrt{\frac{2}{3}(\frac{2}{3}-t)}$		$+\sqrt{\frac{2}{3}(1-t)}$	$\to w$

<u>Proof</u> Using arguments very similar to those of Lemma 1 for each open interval. □

<u>Lemma 3</u> For $t < 0$, $y = -w = +\sqrt{\frac{2}{3}(1-t)}$ we have a downward cusp on v-axis at $v = -w^3 - 2tw$.

<u>Proof</u> As for Lemma 1 except that $y^2 + \frac{2}{3}(t-1) < 0$ in the neighbourhood $-w - \epsilon < y < -w$. This changes over the sign of x corresponding to positive u, and hence we change the direction of the cusp. □

<u>Lemma 4</u> For $t < 0$, $y = -w = \pm\sqrt{(-2t/3)}$ we have a downward pointing angle of magnitude $\pi - 2\tan^{-1}(\sqrt[4]{-6t})$ between the symmetric branches of the curve.

<u>Proof</u> In the y-neighbourhood we have from (13) that

$$\left\{\frac{dv}{du}\right\}^2 = -\frac{27}{8}\,(y + \sqrt{-2t/3})\,(y^2 + 2t/3 - 4/9)$$

$$\to 3\sqrt{-2t/3} = \sqrt{-6t}\ , \text{ as } y \to +\sqrt{-2t/3} = -w.$$

Thus $(dv/du) \to \pm\sqrt[4]{-6t}$ as $y \to -w$ and we get the required angle if y gives real values in the neighbourhood of $y = -w$. This is clear from Fig. 1 □

<u>Definition 5</u> The points $u = 0$, $v = -w^3 - 2tw$, $w = +\sqrt{(-2t/3)}$, $t < 0$ are to be called <u>hyperbolic points</u> (cf. Appendix I).

<u>Lemma 6</u> For $t < 0$, $y = -w = -\sqrt{(-2t/3)}$ we have an isolated real point at $u = 0$, $v = -w^3 - 2tw$.

Proof $y = -w$ gives a single point unless $y = -w = \pm\sqrt{\frac{2}{3}(\frac{2}{3}-t)}$. Fig. 1 shows that neighbouring y-values give imaginary u,v values. $\square$

Definition 7 The points $u = 0$, $v = -w^3 - 2tw$, $w = -\sqrt{(-2t/3)}$, $t<0$ are to be called elliptic points (cf. Appendix I).

Lemma 8 For $t<0$, $y = -w = -\sqrt{\frac{2}{3}(1 - t)}$ we have an upward pointing cusp.

Proof. As for Lemma 3 except that $y<-w$ now gives $y^2 > -\frac{2}{3}(t-1)$ i.e. $y^2 + \frac{2}{3}(t-1) > 0$ so that we are back with situation of Lemma 1. Hence upward pointing cusp. $\square$

Lemma 9 The points $y = -w = \pm\sqrt{\frac{2}{3}(\frac{2}{3}-t)}$ give a branch with downward pointing cusp at $u = 0$, $v = -w^3 - 2tw$.

Proof Clear from (8) and (9). $\square$

The point $y = -w = +\sqrt{\frac{2}{3}(\frac{2}{3}-t)}$ just represents a change in direction of parametrisation from $-w<\sqrt{\frac{2}{3}(\frac{2}{3}-t)}$, where $y \nearrow$ gives $u,v \to \infty$, to the case $-w>\sqrt{\frac{2}{3}(\frac{2}{3}-t)}$ where $y \searrow$ gives $u,v \to \infty$. The behaviour at $y = -w = -\sqrt{\frac{2}{3}(\frac{2}{3}-t)}$ is more complicated since Fig. 1 shows we have real points defined for both $-w + \varepsilon < y < -w$ and $-w - \varepsilon < y < -w$ and the relationship between these points and the semicubic parabola given by (8) and (9) needs some more work.

Lemma 10 For $t<0$, $-w = -\sqrt{\frac{2}{3}(\frac{2}{3}-t)}$ then we get

$$\lim_{y\to -w} u = \pm(4/27w)^{\frac{1}{2}} = u^*$$

$$\lim_{y\to -w} v = -w^3 - 2tw + 4/27w = v^*$$

which corresponds to the points $x = \pm(4/27w)^{\frac{1}{2}}$ in (8) and (9). Further, the tangent direction on the y-parameter curve given by (12) satisfies

$$\lim_{y\to -w} \left\{\frac{dv}{du}\right\} = \frac{2}{3(4/27w)^{\frac{1}{2}}}$$

and hence we get a pair of beak to beak cusps at (u^*,v^*).

Proof To find u^*, v^* we just cancel out the linear factors corresponding to $(y + w) = (y + \sqrt{\frac{2}{3}(\frac{2}{3}-t)})$ and then substitute for y, using the relationship between w and t, to get in required form. Similarly for $\lim_{y\to -w}\left\{\frac{dv}{du}\right\}$ except that we need Assumption A to justify the limit.

To show that this limiting tangent direction is the same as the x - parameter curve we need only differentiate (8) and (9) parametrically to give

$$\frac{dv}{du} = \frac{2}{3x} \qquad x \neq 0$$

and result follows. To show that this gives beak to beak point we can consider the change in v-value for $y = -w \pm \varepsilon$ as compared with v^*. This gives

$$\delta v = v(y \pm \varepsilon) - v^* \simeq \pm \left[\tfrac{2}{3}\left(1 + \frac{1}{4w^2}\right)\right]\varepsilon \qquad \text{- - - - - - - - - - (14)}$$

for first order terms in ε, so that the beak to beak point is established for all $w \neq 0$, i.e. $t < 2/3$. □

The transition involved $w < -\sqrt{\frac{2}{3}(\frac{2}{3} - t)} \longrightarrow w > -\sqrt{\frac{2}{3}(\frac{2}{3} - t)}$ relating to the beak to beak point of Lemma 10 is shown in Fig. 3. We have thus dealt with the line $y = -w$ in Fig. 1 and we now need to consider the other $u = 0$ points, given by $y = \pm\sqrt{-2t/3}$ and $y = \pm\sqrt{\frac{2}{3}(1 - t)}$, $w < \pm\sqrt{\frac{2}{3}(1 - t)}$.

Lemma 11 For $t < 0$, $y = \pm\sqrt{-2t/3} \neq -w$, we have the point $u = 0$, $v = \pm\frac{4t}{3}\sqrt{\frac{-2t}{3}}$, where the symmetric curve has tangent parallel to the u-axis.

Proof The coordinates come from direct substitution; the horizontal tangent comes most easily from (13). This gives

$$\left\{\frac{dv}{du}\right\}^2 \to 0 \quad \text{as } y \to \pm\sqrt{-2t/3}\,. \qquad \square$$

Lemma 12 For $t < 0$, $y = \pm\sqrt{\frac{2}{3}(1-t)} < -w$ we have a real double point at $u = 0$, $v = \pm\frac{4}{3}(t - 1)\sqrt{\frac{2}{3}(1-t)} - 2w$, with tangents at angles,

$$\pm\tan^{-1}\left[\left(\frac{1}{2(\mp\sqrt{2(1-t)/3} - w)}\right)^{\frac{1}{2}}\right]$$

to the u-axis. For $y = \pm\sqrt{2(1-t)/3} > -w$ we have an isolated real point corresponding to a pair of complex critical points.

Proof We have already established that (5) gives complex values of x for the value $y = \pm\sqrt{2(1-t)/3} > -w$. But for this value of y $u = x\left[2(y + w) + x^2\right] = 0$ and we also get $v = \pm\frac{4(t-1)}{3}\sqrt{\frac{2(1-t)}{3}} - 2w$.

For $\pm\sqrt{2(1-t)/3} < -w$ we have from (13) that

$$\left\{\frac{dv}{du}\right\}^2 = -\frac{1}{2(y + w)} = -\frac{1}{2(\pm\sqrt{2(1-t)/3} + w)}$$

and this gives the tangent directions. Further it is not difficult to check that the direction given by positive square root corresponds to the positive sign for u in (6). □

Remark This double point $\rightarrow$ cusp as $-w \neq \sqrt{2(1-t)/3} \longrightarrow 0$

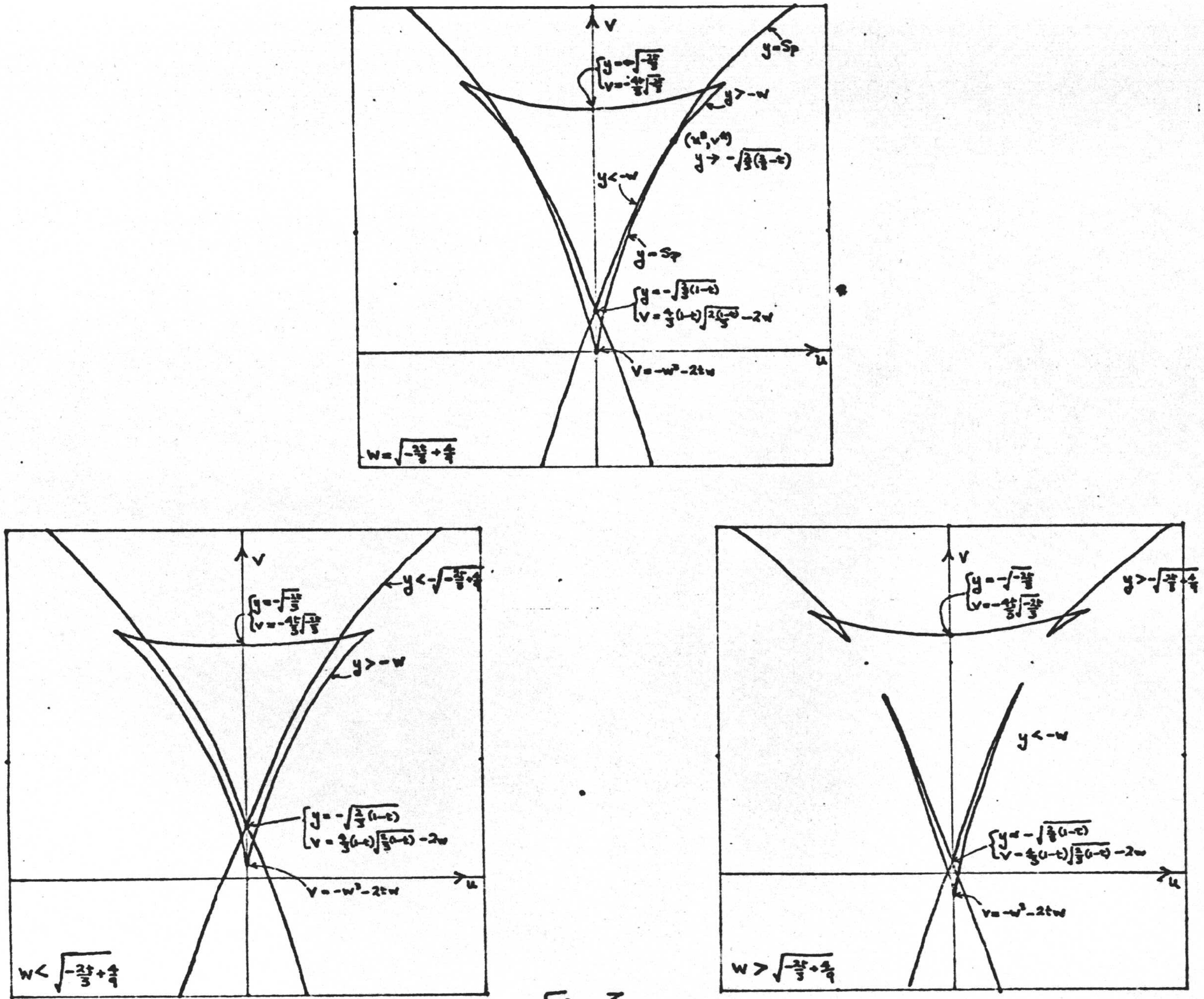


Fig 3.

<u>Lemma 12A</u> For $0 > t > -\frac{1}{3}$ the points $(0, \pm v_1)$ given by $y = -w = \pm\sqrt{\frac{2}{3}(1-t)}$ satisfy the condition $|v_1| > |v_2|$ where the points $(0, \pm v_2)$ are given by $y = \pm\sqrt{-2t/3}$. Similarly for $t = -\frac{1}{3}$ $|v_1| = |v_2|$ and for $t < -\frac{1}{3}$, $|v_1| < |v_2|$.

<u>Proof</u> $y = \pm\sqrt{-\frac{2}{3}t}$ gives $v_2 = \pm\frac{4t}{3}\sqrt{-\frac{2}{3}t}$ and $|v_2| = \frac{-4t}{3}\sqrt{-\frac{2}{3}t}$.

$$y = -w = \pm\sqrt{\tfrac{2}{3}(1-t)} \text{ gives } v = \mp\frac{4}{3}(1-t)\sqrt{\tfrac{2}{3}(1-t)} \pm 2\sqrt{\tfrac{2}{3}(1-t)}$$

$$= \pm\frac{(4t+2)}{3}\sqrt{\tfrac{2}{3}(1-t)} .$$

Squaring we get $|v_2|^2 = \frac{-32}{27}t^3$

and $|v_1|^2 = \frac{-32}{27}\left(t^3 - \frac{3t}{4} - \frac{1}{4}\right)$.

Thus $|v_1|^2 - |v_2|^2 = \frac{8}{27}(3t+1)$ and result follows. □

We have in results 1-12 shown the properties relating to the intersection with $u = 0$ for Fig. 1. In Fig. 2 we see that the diagrams simplify as t increases. The results concerning points of Fig. 1 can easily be shown to hold where they apply in these simpler pictures e.g. $y = -w > \sqrt{2(1-t)/3}$ has an upward cusp all $t < 1$. This is just a matter of checking the proofs of results 1-12. However, new features relating to v-axis are introduced at transition t values $t = 0, 2/3, 1$. We must then consider these special points.

<u>Lemma 13</u> For $t = 0$, $y = w = 0$ gives the point $u = v = 0$ on a symmetric curve having the u-axis as tangent line.

<u>Proof</u> For $w = t = 0$ we use (13) to give

$$\left\{\frac{dv}{du}\right\}^2 = \frac{-27}{8}y^3$$

which establishes the tangent direction. Direct substitution gives $u = v = 0$. □

<u>Definition 13A</u> The point $w = t = u = v = 0$ will be called a <u>parabolic point</u>.

<u>Lemma 14</u> For $t = 2/3$, $w = 0$; $y = 0$ is a special point giving rise to the semi-cubic parabola $u = x$, $v = x$. But the beak to beak points of Lemma 10 are at infinity.

<u>Proof</u> For $t = 2/3$, $w = 0$ (4) reduces to

$$9x^2 y^2 + 2y\left(3y^2 + \frac{4}{3}\right) = 0$$

so that $y = 0$ gives no condition on x and we have

$$u = (2(0+0) + x^2)\,x = x^3 ;$$
$$v = 0^2 + 2.\tfrac{2}{3}.0 + x^2 = x^2 .$$

To show that beak to beak point at infinity we consider $\lim\limits_{y \to 0-} u$, $\lim\limits_{y \to 0-} v$

i.e.

$$\lim_{y \to 0-} \left(\frac{4(y^2 - {}^2\!/_9)}{3y} \sqrt{\frac{-2(y^2 + {}^4\!/_9)}{3y}} \right) = \infty$$

$$\lim_{y \to 0-} \left(y(y^2 + \frac{4}{3}) - \frac{2(y^2 + {}^4\!/_9)}{3y} \right) = \infty$$

which shows that point corresponding to v^*, u^* in Lemma 10 is not a finite point of the plane. $\square$

Lemma 15 For $t = 1$, $w = 0$; $y = 0$ gives a symmetric upward pointing cusp at $u = v = 0$. For $t = 1$, $w < 0$ we have that $y = 0$ gives $u = 0$, $v = -2w$ which the common point of symmetrically placed upward pointing cusps with axes on the lines with gradients $\pm\left(-\frac{1}{2w}\right)^{\frac{1}{2}}$. (A bent beak to beak point).

Proof For $w = 0$ we have the same argument as Lemma 8, but for $w < 0$ we need first to establish the fact that $y = 0$ gives an 'off-axis' cusp from $3y^2 + 2t + 2x\frac{dx}{dy} = 0$ and this will be done in §4. Once this is known, we can use (13) to find the direction of the axis of the cusp, i.e.

$$\left\{\frac{dv}{du}\right\}^2 = -\frac{27}{8} \frac{(y^2 + \frac{2}{3})(y^2 + \frac{2}{9})}{(y + w)} \to -\frac{1}{2w} \text{ as } y \to 0.$$

and then use find the u,v words by direct substitution. Lastly we need to check the signs of u and $\frac{dv}{du}$ to place the cusps precisely. $\square$

This completes the analysis of the intersections with the v-axis and some related points arising in the special $-w = \pm\sqrt{\frac{4}{3}\left(\frac{2}{3} - t\right)}$ cases. These results will be used in drawing the u,v cross-sections given in Figs. 6-12.

§ 4 Detailed Analysis - Cusps

The cusps on the u,v curves are given by $\frac{dv}{dy} = \frac{du}{dy} = 0$ and this gives from (11) and (12) the equation

$$3y^2 + 2t + 2x\frac{dx}{dy} = 0 \qquad \text{- - - - - - - - - - - - - - -} \quad (15)$$

From (5) we have

$$x\frac{dx}{dy} = -\frac{\left(y^4 + \frac{4}{3}(t - 1)y^2 - \frac{8w}{9}y + \left(\frac{4t}{9} - \frac{8t}{27}\right)\right)}{3\,(y^2 + {}^t\!/_3 - 4/9)^2} \qquad \text{- - -} \quad (16)$$

hence the y equation for cusp points is

$$y^6 + 2(t - {}^5\!/_9)y^4 + \frac{4}{3}\left\{t - \frac{10t}{9} + \frac{10}{27}\right\}y^2 + \frac{16w}{81}y + \frac{8t}{27}(t-1)(t-\tfrac{2}{3}) = 0 \qquad (17)$$

assuming $(y^2 + 2t/3 - 4/9) \neq 0$. (This shows that Assumption A is justified).

To investigate the solutions of (17) we consider the problem in terms of the intersection of the two curves

$$z = y^6 + 2(t - \tfrac{5}{9})y^4 + \tfrac{4}{3}(t^2 - \tfrac{10t}{9} + \tfrac{10}{27})y^2 + \tfrac{8t}{27}(t - 1)(t - \tfrac{2}{3}) \qquad (18)$$

$$z = -\tfrac{16w}{81}y. \qquad (19)$$

The first stage of the investigation is the nature of the curve given by (18).

Lemma 16 The equation

$$y^6 + 2(t - \tfrac{5}{9})y^4 + \tfrac{4}{3}(t^2 - \tfrac{10t}{9} + \tfrac{10}{27})y^2 + \tfrac{8t}{27}(t - 1)(t - \tfrac{2}{3}) = 0$$

has (i) no real root if $t > 1$ or $0 < t < 2/3$;

(ii) a double root $y = 0$ if $t = 0, 1, 2/3$;

(iii) one pair of real roots $y = \pm\sqrt{c}$ if $t < 0$ or $2/3 < t < 1$.

Proof We consider the equation as a cubic in y^2 and then it expresses it in the reduced form

$$\left(y^2 + \tfrac{2}{3}(t - \tfrac{5}{9})\right)^3 + A\left(y^2 + \tfrac{2}{3}(t - \tfrac{5}{9})\right) + B = 0$$

Calculation gives $A = 20/243$ so that we can have only one real root for y^2. If we then look at the product of the roots for the original equation we see that this real root for y^2 is ; positive if $\tfrac{8t}{27}(t - 1)(t - \tfrac{2}{3}) < 0$; zero if $\tfrac{8t}{27}(t - 1)(t - \tfrac{2}{3}) = 0$; negative if $\tfrac{8t}{27}(t - 1)(t - \tfrac{2}{3}) > 0$. The result then follows. $\square$

Lemma 17 The curve given by (18) has only one stationary point and that is a minimum at $y = 0$.

Proof $$\frac{dz}{dy} = 6y\left[(y^2 + \tfrac{2}{3}(t - \tfrac{5}{9}))^2 + \frac{20}{729}\right] \qquad (20)$$

Clearly $\frac{dz}{dy} > 0$ for $y > 0$; $\frac{dz}{dy} = 0$ if $y = 0$ and $\frac{dz}{dy} < 0$ if $y < 0$ hence we get result. $\square$

Lemma 18 The curve given by (18) has 4 inflexion points if $t < 5/18$; 2 inflexions if $t = 5/18$ and none otherwise.

Proof $$\frac{d^2z}{dy^2} = 30\left[y^4 + \tfrac{4}{5}(t - \tfrac{5}{9})y^2 + \tfrac{4}{45}(t^2 - \tfrac{10}{9}t + \tfrac{10}{27})\right] \qquad (21)$$

To find $\frac{d^2z}{dy^2} = 0$ we need to solve a quadratic in y^2 and we get

$$y^2 = -\tfrac{2}{5}(t - \tfrac{5}{9}) \pm \tfrac{4}{15}\sqrt{(t - \tfrac{5}{18})(t - \tfrac{5}{6})}$$

which has real roots if $t \geqslant 5/6$ or $t \leqslant 5/18$. The product of the roots for y^2 is positive for all t since

$$t^2 - \frac{10}{9}t + \frac{10}{27} = (t - \frac{5}{9})^2 + \frac{5}{81} .$$

so when real we have both positive or both negative. Now the sum of the roots is $-4/5\,(t - 5/9)$ so that we get that both real roots for y^2 are positive if $t \leqslant 5/18$ and both negative if $t \geqslant 5/6$. Hence result. □

From the results of Lemma 16-18 and the symmetry with respect to y it is not difficult to sketch the graphs for (18) for various t. We see that to solve the equation (17) we need to draw the straight lines through the origin given by (19) and find their intersections with the graphs of (18). This will give a rough idea of the number of cusps for a particular w,t pair and we can see how we can expect the y values giving cusps to vary with w. The important stages in such variations of w will be the cases where the line given by (19) is tangent to the curve given by (18). To be able to deduce the nature of the variation of the y with w, it is sufficient to know the number of such transition points for a given t. The equation for y values of such transition points is seen geometrically to be

$$(18)/_y = \frac{d}{dy}\,[\,(18)\,] .$$

which gives on simplification

$$y^6 + \frac{6}{5}(t - \frac{5}{9})y^4 + \frac{4}{15}(t^2 - \frac{10}{9}t + \frac{10}{27})\,y^2 - \frac{8t}{135}(t - 1)(t - \frac{2}{3}) = 0 \quad -- (22)$$

__Lemma 19__ As a cubic in y^2, (22) has one real root for $\frac{5}{54} < t < 5/6$ and three real roots otherwise.

__Proof__ We first express (22) in the reduced cubic form

$$(y^2 + \frac{2}{5}(t - \frac{5}{9}))^3 + A(y^2 + \frac{2}{5}(t - \frac{5}{9})\ ) + B = 0$$

and find

$$A = -\frac{16}{75}\,(t - \frac{5}{6})\,(t - \frac{5}{18})$$

$$B = -\frac{128t}{27.125}\,(t - \frac{5}{6})^2$$

and then show that the discriminant of the cubic is

$$-\frac{3.25}{(18)^2}\;\frac{(128)^2}{(27)^2\,(125)^2}\,(t - \frac{5}{54})\,(t - \frac{5}{6})^3 .$$

Thus the result follows from the general rule about the sign of the discriminant. For the values $t = 5/6$ and $t = 5/54$ we have a pair of coincident roots. □

__Lemma 20__ In the case where (22) has one real root for y^2, then this root is positive if $5/54 < t < 2/3$; zero if $t = 2/3$ and negative if $2/3 < t \leqslant 5/6$.

<u>Proof</u> The product of the roots of (22) is $\frac{8t}{135}(t - 1)(t - 2/3)$ and with one real root the sign of this expression determines the sign of the root. With the results of Lemma 19 to limit the range of t the result follows from this product. □

<u>Lemma 21</u> If $t < 0$ then (22) has one negative and two positive roots for y^2.

<u>Proof</u> For $t < 0$ the product of the roots is negative, hence either one or three negative roots. But the sum of the roots given by $-6/5(t - 5/9)$ is positive so that we cannot have three negative roots. □

<u>Lemma 22</u> If $t > 1$ then (22) has one positive and two negative roots.

<u>Proof</u> For $t > 1$ the product the roots is positive, so that we have either one positive or three positive roots. But sum of roots is negative so that at least one negative root, hence result. □

<u>Lemma 23</u> If $t = 0$ then (22) has one zero root and two positive roots.

<u>Proof</u> Product of distinct roots (see Lemma 19) zero hence one zero root. Sum of roots positive is not conclusive, but the sum of products two at a time being $40/(15 \times 27) > 0$ shows that both are the same sign, hence positive. □

<u>Lemma 24</u> If $t = 1$ then (22) has one zero root and two negative roots.

<u>Proof</u> Similar to 23. □

<u>Lemma 25</u> If $0 < t \leqslant 5/54$ then (22) has three positive roots, distinct except at $t = 5/54$ where we have a double root and a simple root.

<u>Proof</u> If the roots are $r_1(t)$, $r_2(t)$, $r_3(t)$ then they are continuous functions of t for the range of t $(0, 5/54)$. (We could write down the explicit formulae.) From Lemma 23 we can write

$$r_1(0) > r_2(0) > r_3(0) = 0$$

and by continuity we have for sufficient small $t > 0$

$$r_1(t) > r_2(t) > 0 .$$

But the product of the roots is positive hence $r_3(t) > 0$ for the same small range of $t > 0$. By the continuity of the three functions we need the product of the roots to vanish if any of them change sign as t increases. Since the product remains positive, we get the required result for $0 < t \leqslant 5/54$.

At $t = 5/54$ we have at least two of the real roots coincident. There are only two coincident since otherwise $A = \frac{16}{75}(t - \frac{5}{6})(t - \frac{5}{18})$ would have to be zero for all three roots coincident. $\square$

<u>Lemma 26</u> If $1 > t \geq 5/6$ then (22) has three negative roots that are distinct, except in the case $t = 5/6$ where there is a triple root.

<u>Proof</u> Similar to Lemma 25. $\square$

Using the results of Lemmas 16-26 we can show all the cases for the solution of (17). These are given graphically in Fig. 4. However, there remain two problems, what to do about the $y^2 + \frac{2}{3}(t - \frac{2}{3}) \neq 0$ condition and what solutions of (17) correspond to real values of u ? From §2 we know that $y^2 + \frac{2}{3}(t - \frac{2}{3}) = 0$ corresponds to a u,v point at infinity except in the special case $-w = \pm\sqrt{\frac{2}{3}(\frac{2}{3} - t)}$. We can thus ignore the effect of the restriction apart from special cases to be dealt with later. It is possible to deduce for fixed t rough (y,w) graphs from the diagrams of Fig. 4. In order to see how these related to the results of §2 given in Figs. 1-2 we see that we must find out where these cusp lines in the (y,w) plane intersect the line $y = -w$. This gives rise to solving the equation

$$y^6 + 2(t - \frac{5}{9})y^4 + \frac{4}{3}(t^2 - \frac{10}{9}t + \frac{2}{9})y^2 + \frac{8t}{27}(t - 1)(t - \frac{2}{3}) = 0 \quad ---- \quad (23)$$

<u>Lemma 27</u> Equation (23) has roots given by

$$y^2 = -\frac{2t}{3}, \quad y^2 = -\frac{2t}{3} + \frac{4}{9}, \quad y^2 = -\frac{2t}{3} + \frac{2}{3}$$

<u>Proof</u> If we express (23) in the reduced cubic form we get

$$(y^2 + \frac{2}{3}(t - \frac{5}{9}))^3 - \frac{28}{243}(y^2 + \frac{2}{3}(t - \frac{5}{9})) + \frac{160}{27.9^3} = 0$$

so that we get three t independent roots r_1, r_2, r_3 for $y^2 + \frac{2}{3}(t - \frac{5}{9})$. These roots are distinct real roots since the discriminant is given by

$$\frac{2^8}{3^{18}}(100 - 343) = -\frac{2^8}{3^{13}} < 0$$

Thus we can have $r_1 < r_2 < r_3$ and get

$$y^2 = r_1 - \frac{2}{3}(t - \frac{5}{9}), \; r_2 - \frac{2}{3}(t - \frac{5}{9}), \; r_3 - \frac{2}{3}(t - \frac{5}{9})$$

as roots for all values of t.

Since the product of the roots of (23) is $\frac{8t}{27}(t - \frac{2}{3})(t - 1)$ we see that we get a zero root for $t = 0, 2/3, 1$. Putting in these values of t we can thus evaluate r_1, r_2, r_3 as $-10/27, 2/27, 8/27$

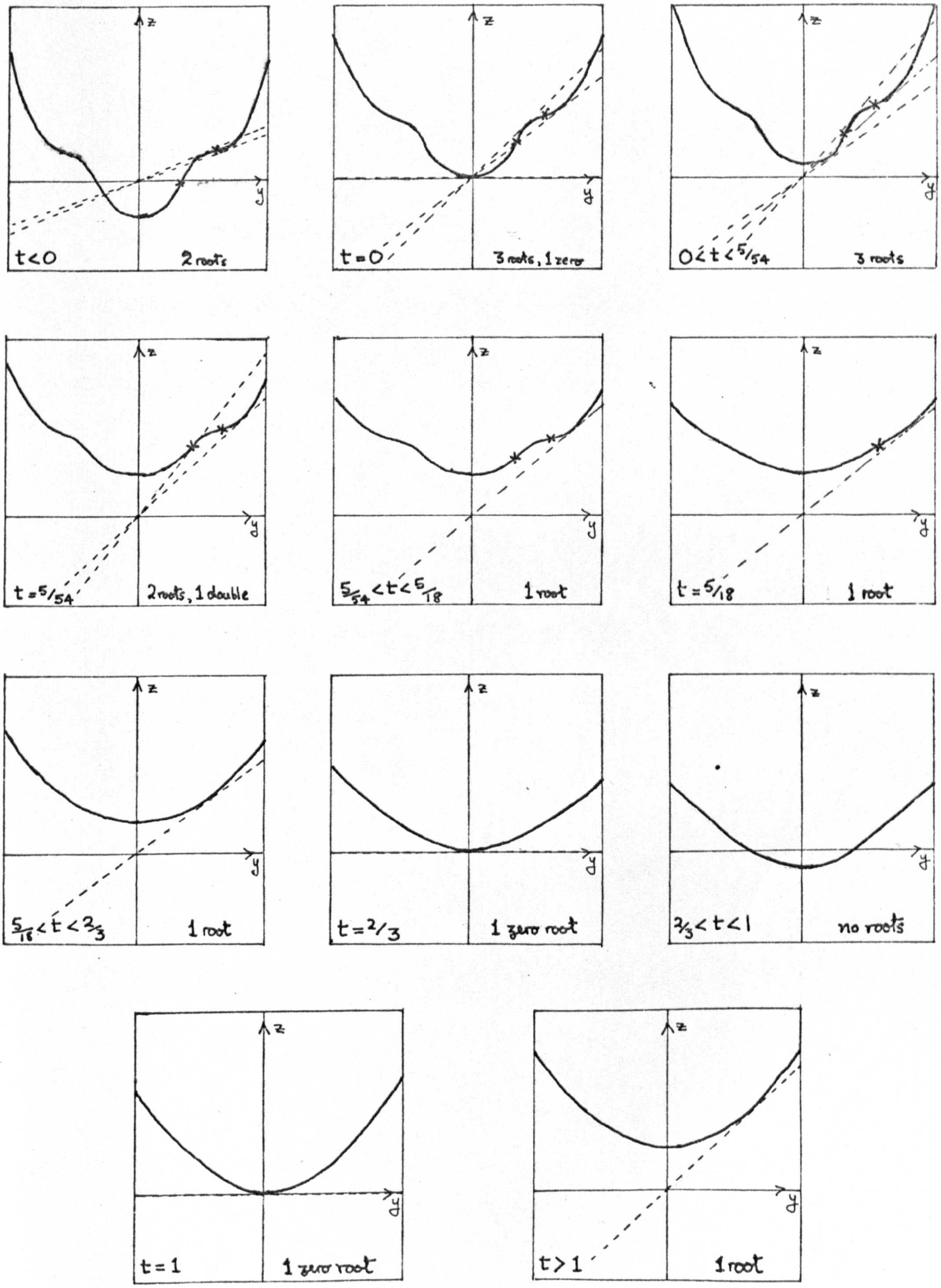

t<0
2 roots
t=0
3 roots, 1 zero
0<t<5/54
3 roots
t=5/54
2 roots, 1 double
5/54<t<5/18
1 root
t=5/18
1 root
5/18<t<2/3
1 root
t=2/3
1 zero root
2/3<t<1
no roots
t=1
1 zero root
t>1
1 root

Fig 4.

respectively. This gives

$$y^2 = -\frac{2t}{3}, \ -\frac{2t}{3} + \frac{4}{9}, \ -\frac{2t}{3} + \frac{2}{3}$$

as required. $\square$

From this last result and the diagrams of Fig. 4 we can sketch in the cusp line on the diagrams of Fig. 5. It should be pointed out, however, that the swallow tail points and the butterfly point have not been confirmed analytically, since we have only found number of roots for (22) and not the w and y values to which these roots correspond. The equation for the w - values can easily be written in terms of the vanishing of an 11 x 11 determinant, the discriminant of (17), which involves w and t. But this is analytically rather cumbersome so that we are left with results of computer calculations which are only confirmations in terms of particular values of w,t.

§ 5 - The (u,v) - Pictures

Using the basic equations of §1 and the analysis of these equations given in §§2-4, it is possible to get reasonable sketches of the u,v graphs for any given w,t. However, the analysis of the parameter values for cusps has not been completed and no attempt has been made to calculate any double points due to intersection of different branches (cf. Appendix I where such an analysis for double points was done). Using a computer with a graph plotter it is possible to draw more accurate pictures using the basic equations and hence get for particular values of w,t an indication of the existance of double points. It should be pointed out, however, that particularly in the neighbourhood of $y = -w = -\sqrt{\frac{2}{3}(\frac{2}{3} - t)}$ $(t<0)$ it requires a fairly sophisticated program to plot all the features that occur.

Using analytic results and information from computer results we draw the different families of (u,v) graphs corresponding to characteristic values of t. In Figs. 6-12 we give some of these families of graphs together with the t = const. (w,v) graphs which can be used to string together the family to get a t = const, 3-dimensional picture. Also included on these diagrams are the indications of the number and type of critical points for potential functions corresponding to the components of $\mathbb{R}^4 - \{$bifurcation set$\}$. Only the number of maxima and minima has been written down since the number of saddles is always one less than the total of the other two. The results for these numbers was obtained from rough drawings for graphs of (2), (3) and (5).

In Fig. 6 we give in 6a the (w,v) graph deduced from the u = 0

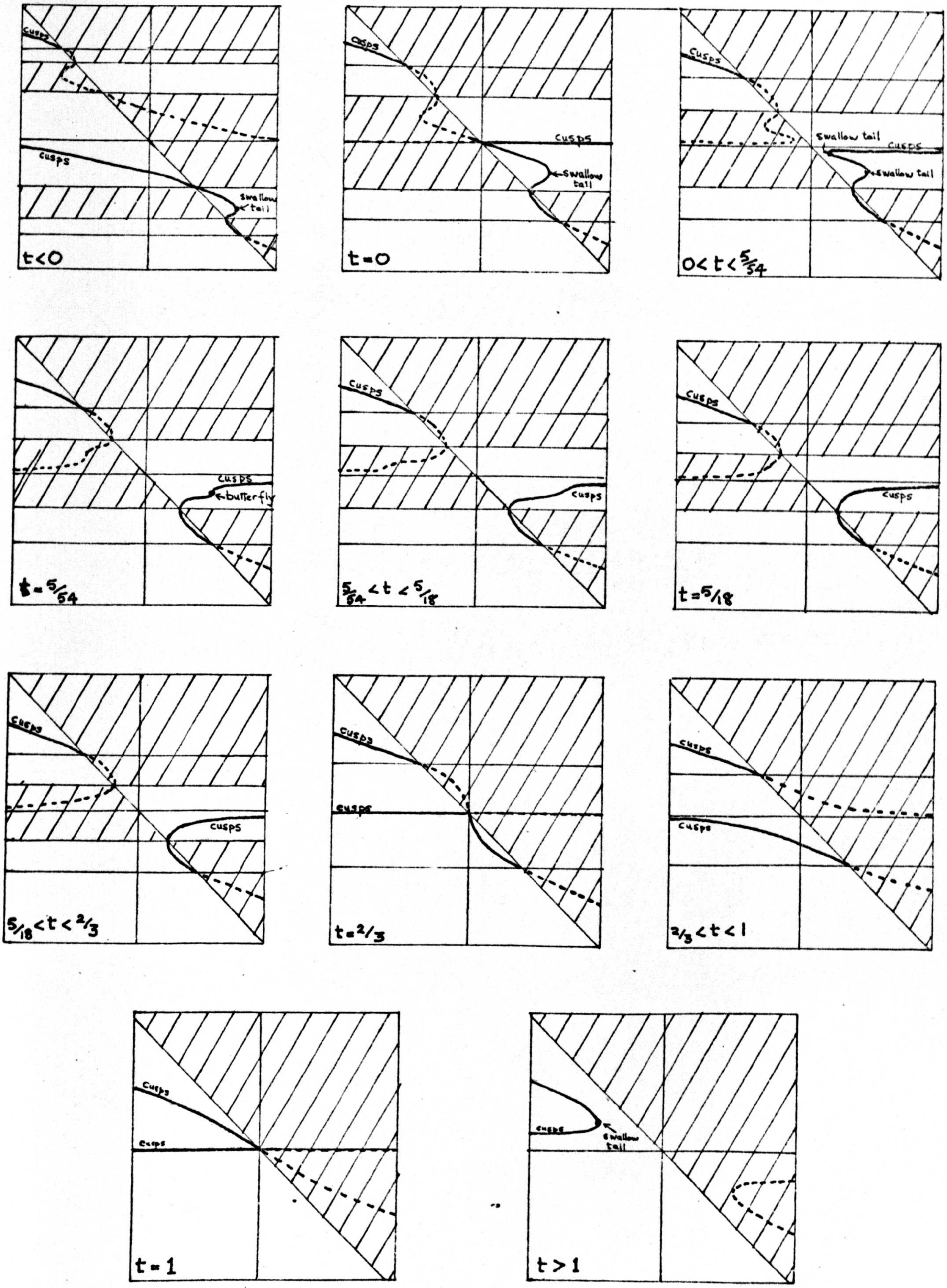
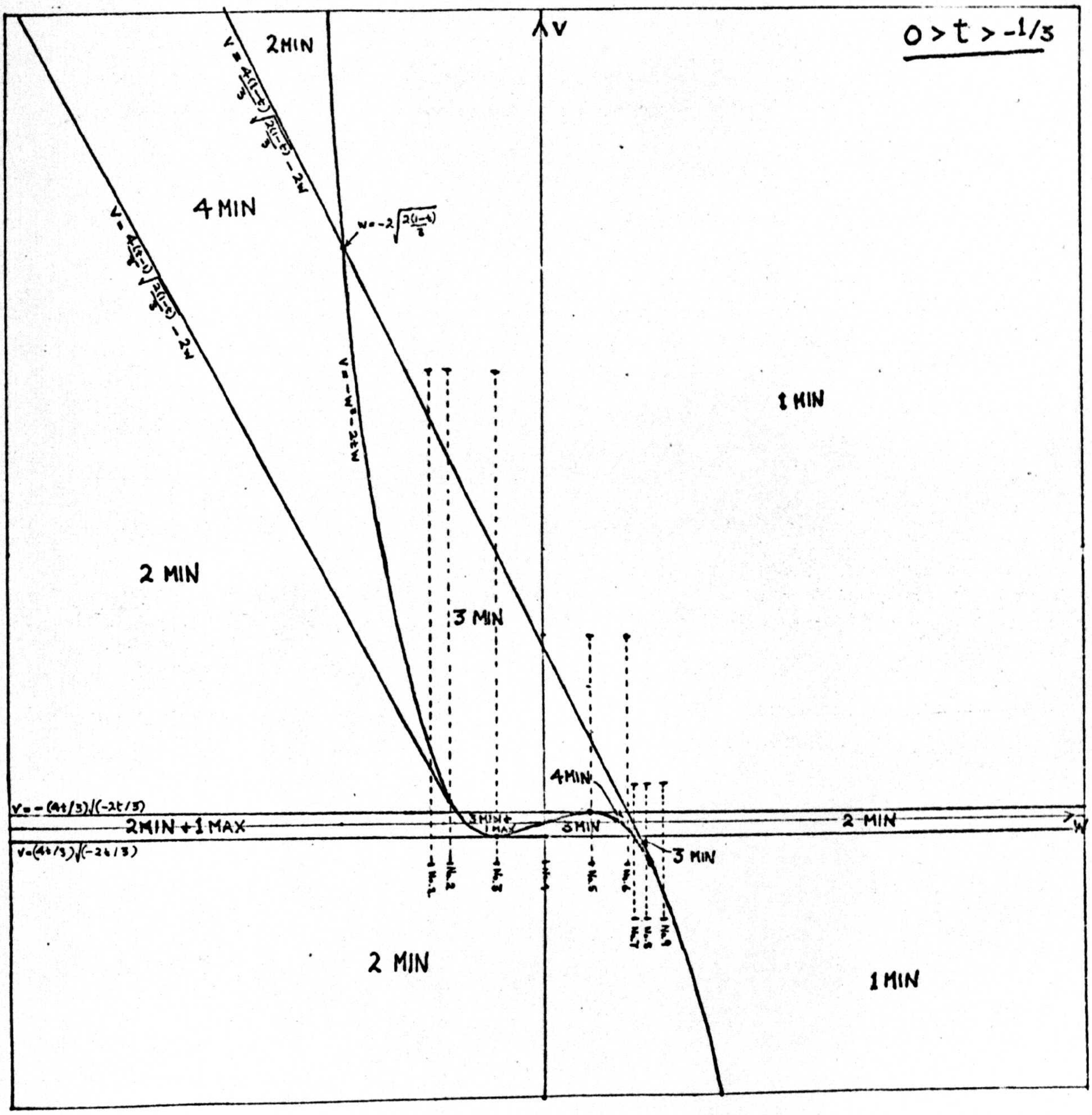

cusps
cusps
swallow tail
t<0
cusps
cusps
swallow tail
t=0
cusps
swallow tail
cusps
swallow tail
0<t<5/54
cusps
butterfly
t=5/54
cusps
cusps
5/54<t<5/18
cusps
cusps
t=5/18
cusps
cusps
5/18<t<2/3
cusps
cusps
t=2/3
cusps
cusps
2/3<t<1
cusps
cusps
t=1
cusps
swallow tail
t>1

Fig 5.



0 > t > -1/3
V
W
2 MIN
4 MIN
2 MIN
3 MIN
1 MIN
4 MIN
2 MIN + 1 MAX
3 MIN + 1 MAX
3 MIN
2 MIN
3 MIN
2 MIN
1 MIN

Fig 6a.

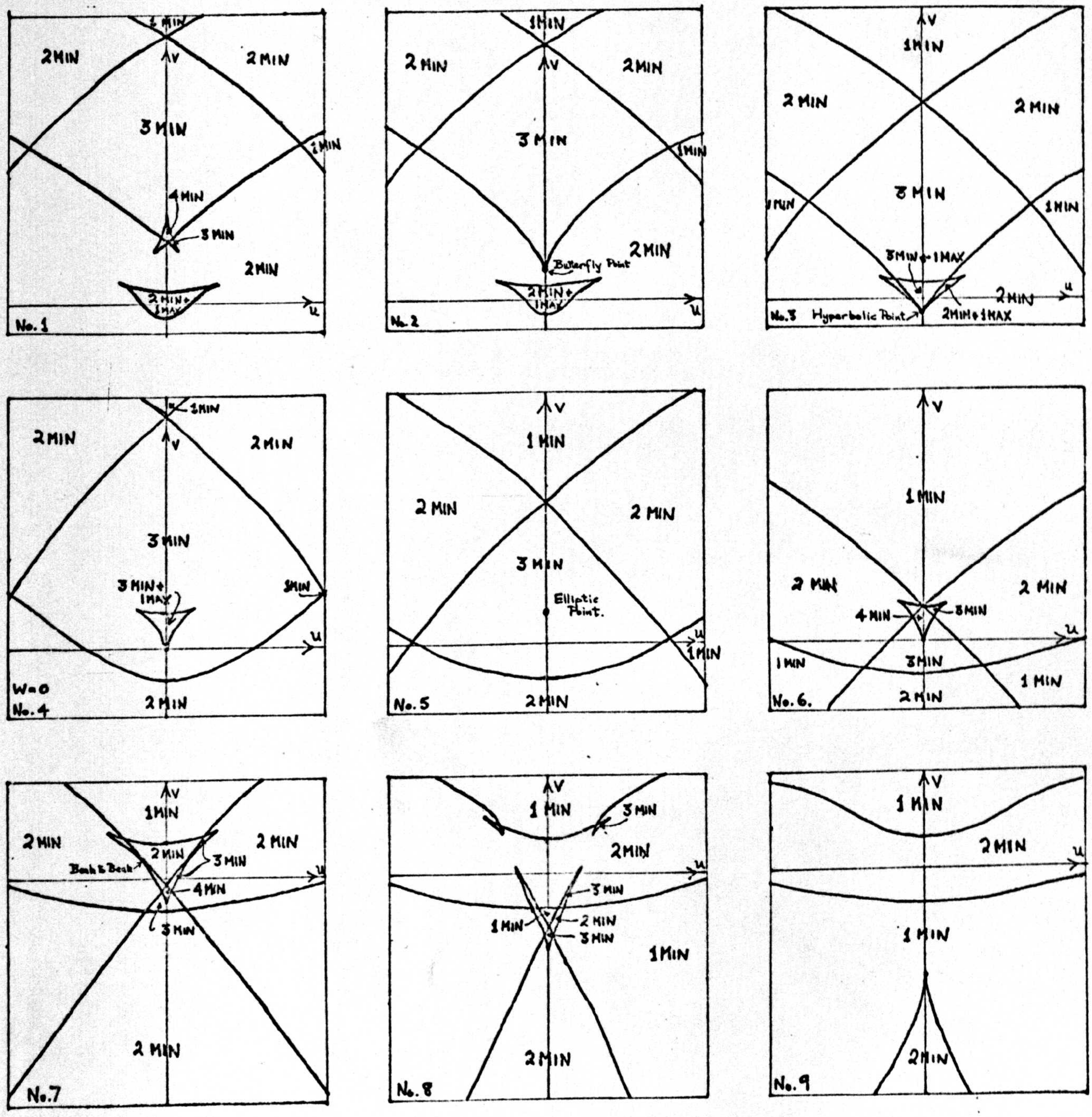

No. 1
2 MIN
3 MIN
1 MIN
4 MIN
2 MIN + 1 MAX
No. 2
Butterfly Point
No. 3
3 MIN + 1 MAX
2 MIN + 1 MAX
Hyperbolic Point
No. 4
W = 0
3 MIN + 1 MAX
No. 5
Elliptic Point.
No. 6.
No. 7
Beak & Beak
No. 8
No. 9
u
v

Fig 6b

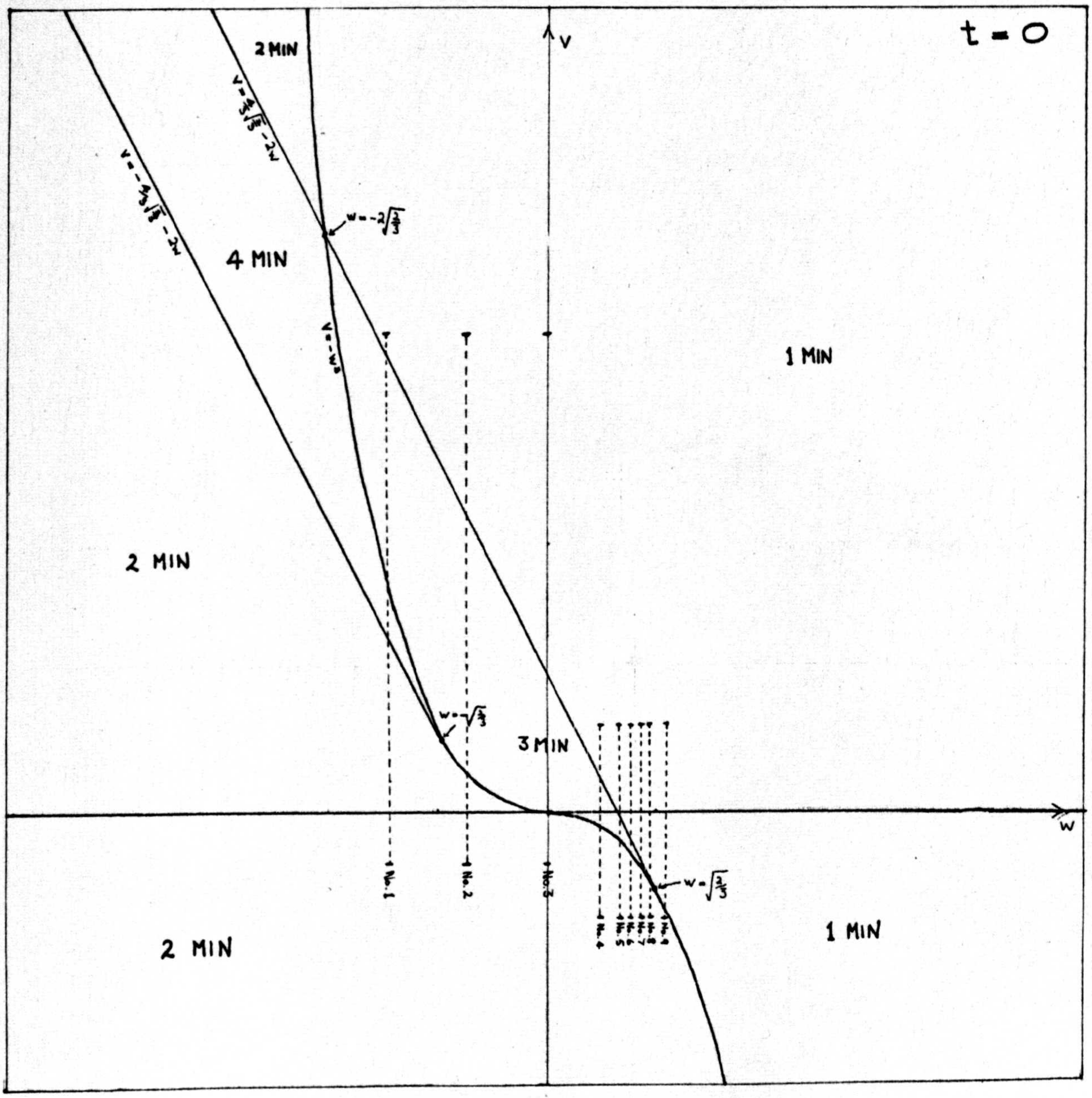
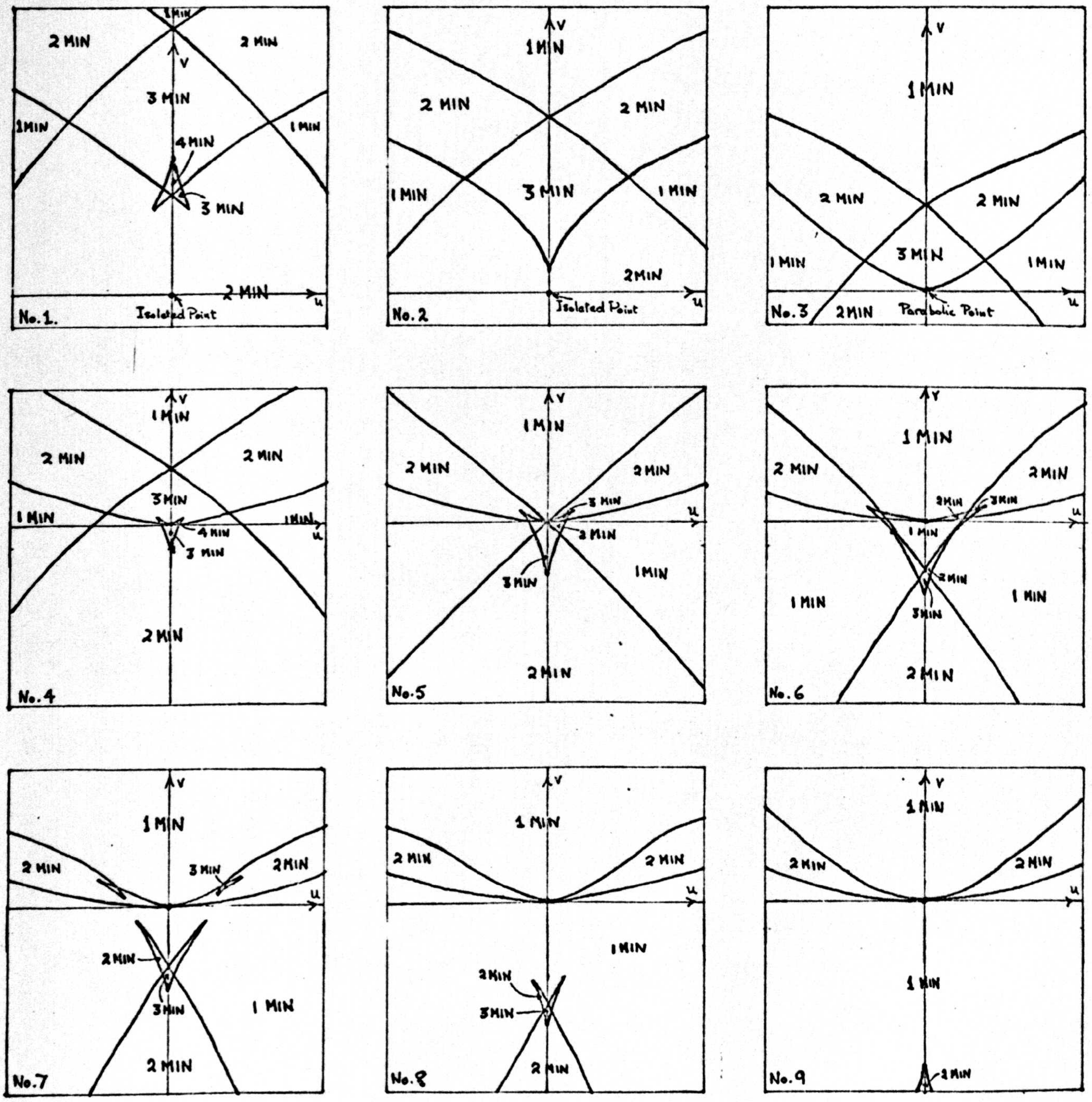


Fig. 7a.



No. 1.
2 MIN
2 MIN
1 MIN
V
3 MIN
1 MIN
1 MIN
4 MIN
3 MIN
2 MIN
u
Isolated Point
No. 2
V
1 MIN
2 MIN
2 MIN
1 MIN
3 MIN
1 MIN
2 MIN
u
Isolated Point
No. 3
V
1 MIN
2 MIN
2 MIN
3 MIN
1 MIN
1 MIN
2 MIN
Parabolic Point
u
No. 4
V
1 MIN
2 MIN
2 MIN
3 MIN
1 MIN
1 MIN
u
4 MIN
3 MIN
2 MIN
No. 5
V
1 MIN
2 MIN
2 MIN
3 MIN
u
2 MIN
3 MIN
1 MIN
2 MIN
No. 6
V
1 MIN
2 MIN
2 MIN
2 MIN
3 MIN
u
1 MIN
2 MIN
1 MIN
1 MIN
3 MIN
2 MIN
No. 7
V
1 MIN
2 MIN
3 MIN
2 MIN
u
2 MIN
3 MIN
1 MIN
2 MIN
No. 8
V
1 MIN
2 MIN
2 MIN
u
1 MIN
2 MIN
3 MIN
2 MIN
No. 9
V
1 MIN
2 MIN
2 MIN
u
1 MIN
1 MIN

Fig. 7b.

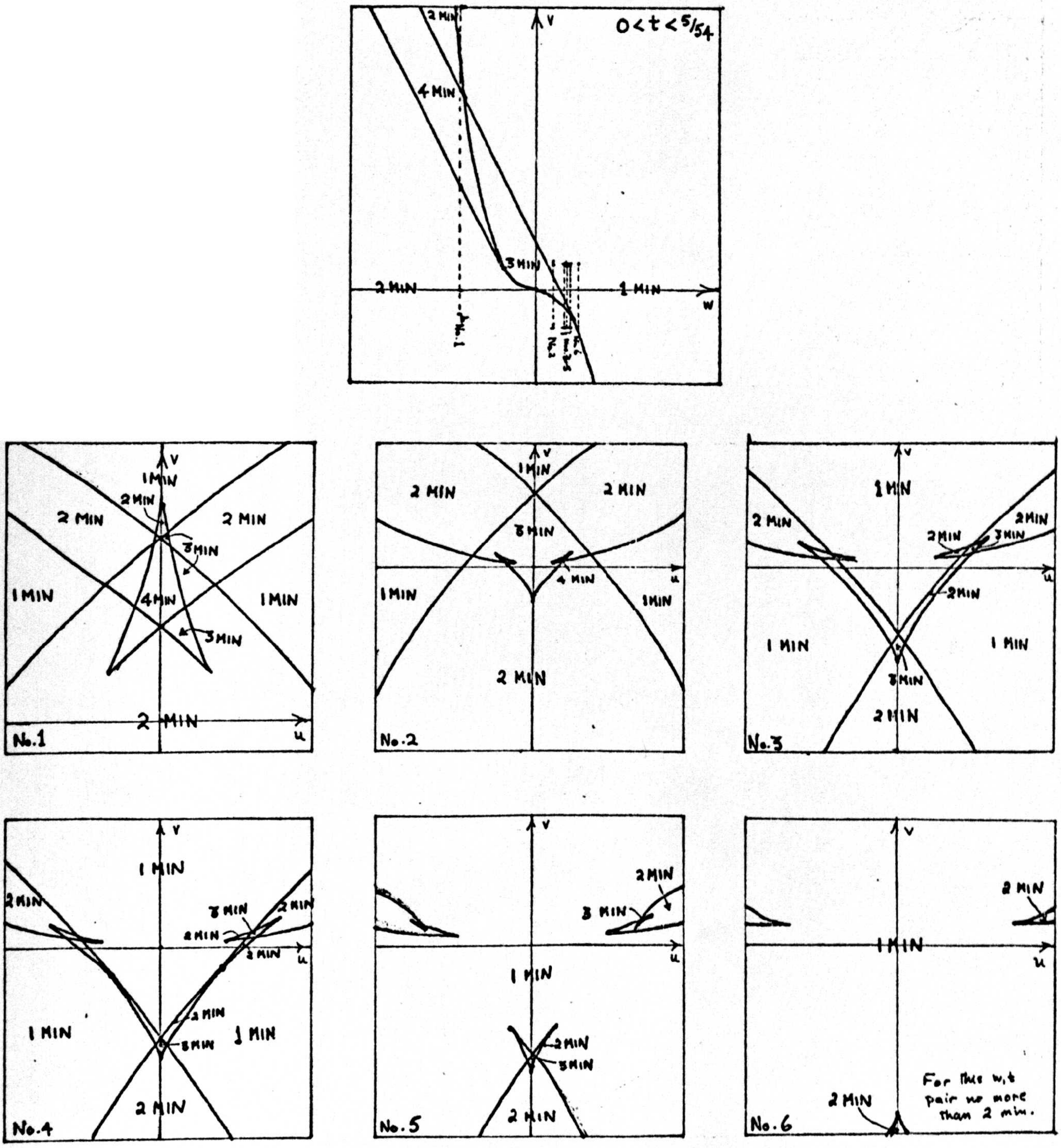

0<t<5/54
2 MIN
4 MIN
3 MIN
1 MIN
v
w
u
No. 1
No. 2
No. 3
No. 4
No. 5
No. 6
3 MIN
4 MIN
For this w,t pair no more than 2 min.

Fig. 8.

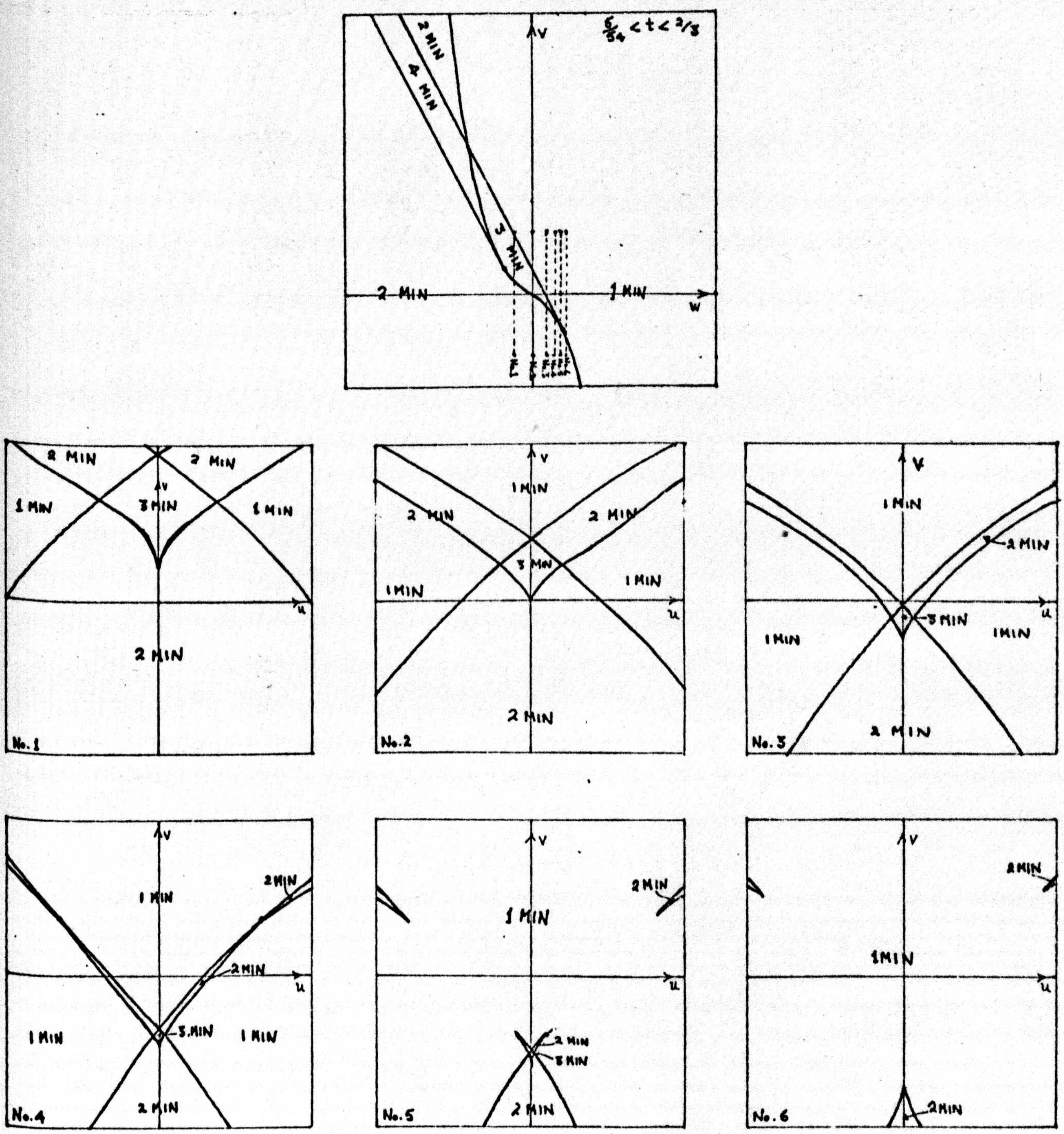

2 MIN
4 MIN
3 MIN
2 MIN
1 MIN
v
w
No. 1
No. 2
No. 3
No. 4
No. 5
No. 6
1 MIN
2 MIN
3 MIN
u

Fig. 9.

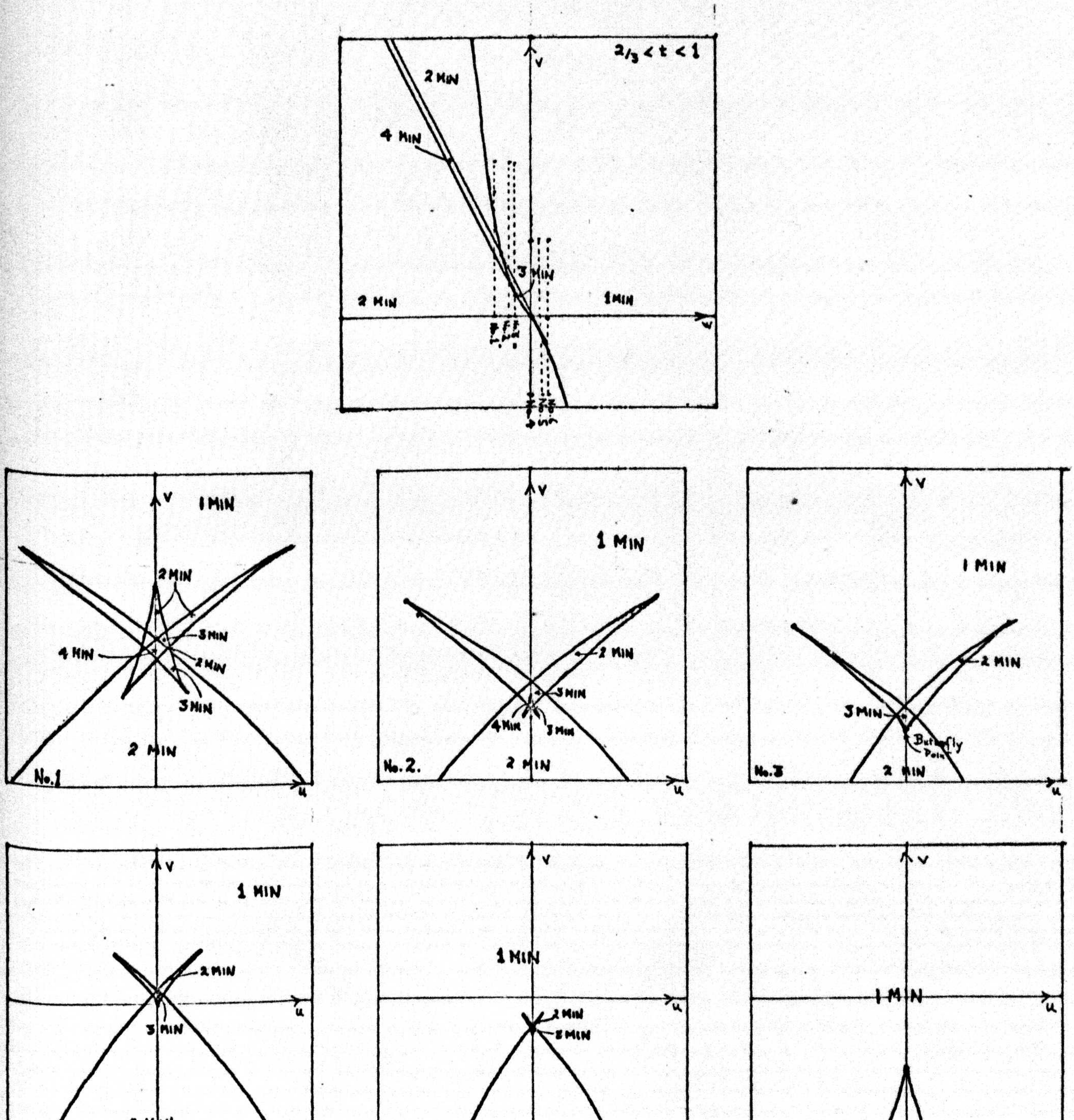

$2/3 < t < 1$
2 MIN
4 MIN
3 MIN
2 MIN
1 MIN
1 MIN
2 MIN
3 MIN
4 MIN
2 MIN
3 MIN
2 MIN
No.1
1 MIN
2 MIN
3 MIN
4 MIN
3 MIN
2 MIN
No. 2.
1 MIN
2 MIN
3 MIN
2 MIN
No.3
1 MIN
2 MIN
3 MIN
2 MIN
1 MIN
2 MIN
3 MIN
2 MIN
No.5
1 MIN
2 MIN
No. 6

Fig. 10.

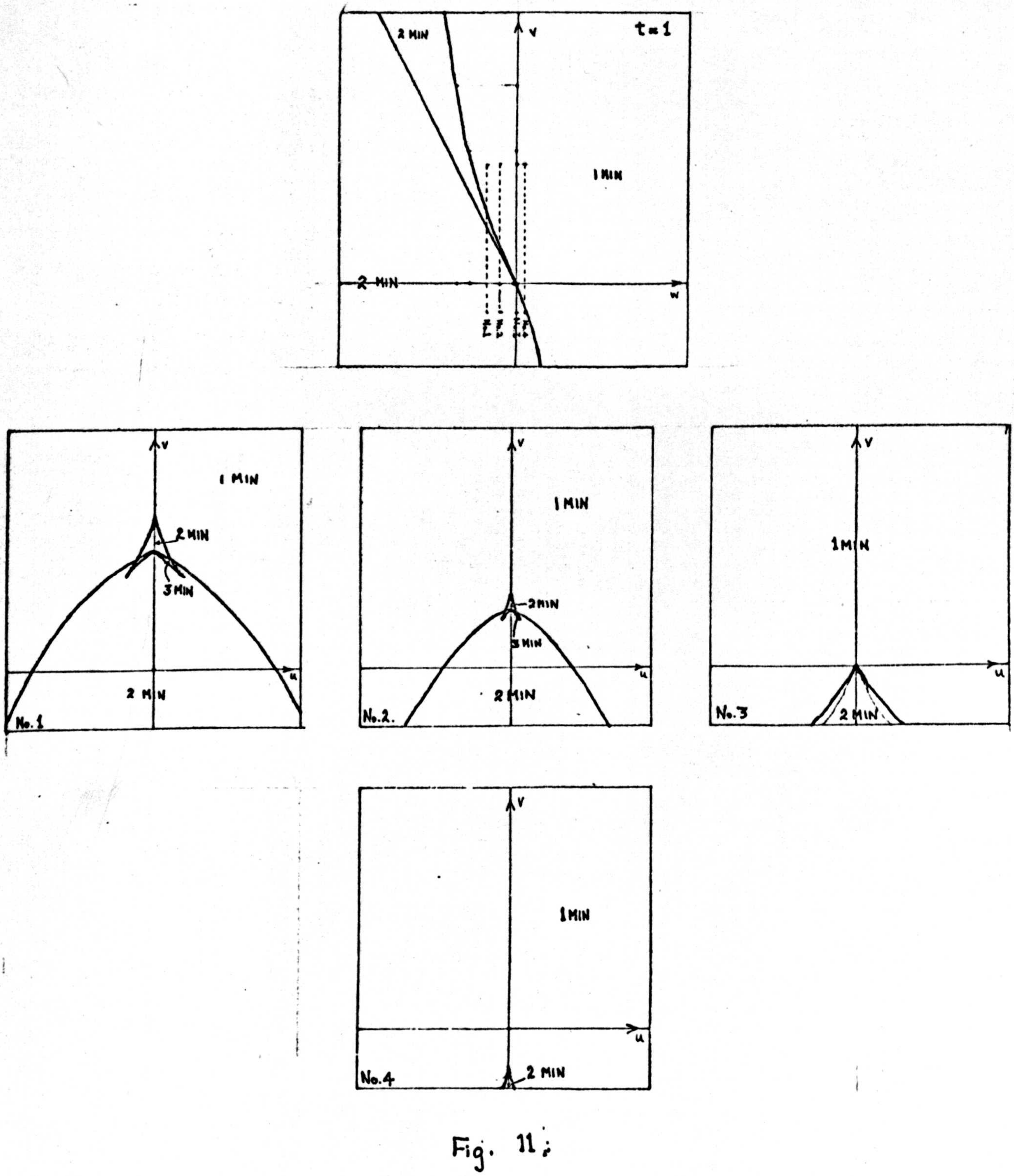

t = 1
2 MIN
1 MIN
2 MIN
v
w
No. 1
1 MIN
2 MIN
3 MIN
2 MIN
No. 2.
1 MIN
2 MIN
3 MIN
2 MIN
No. 3
1 MIN
2 MIN
No. 4
1 MIN
2 MIN
u
v

Fig. 11.

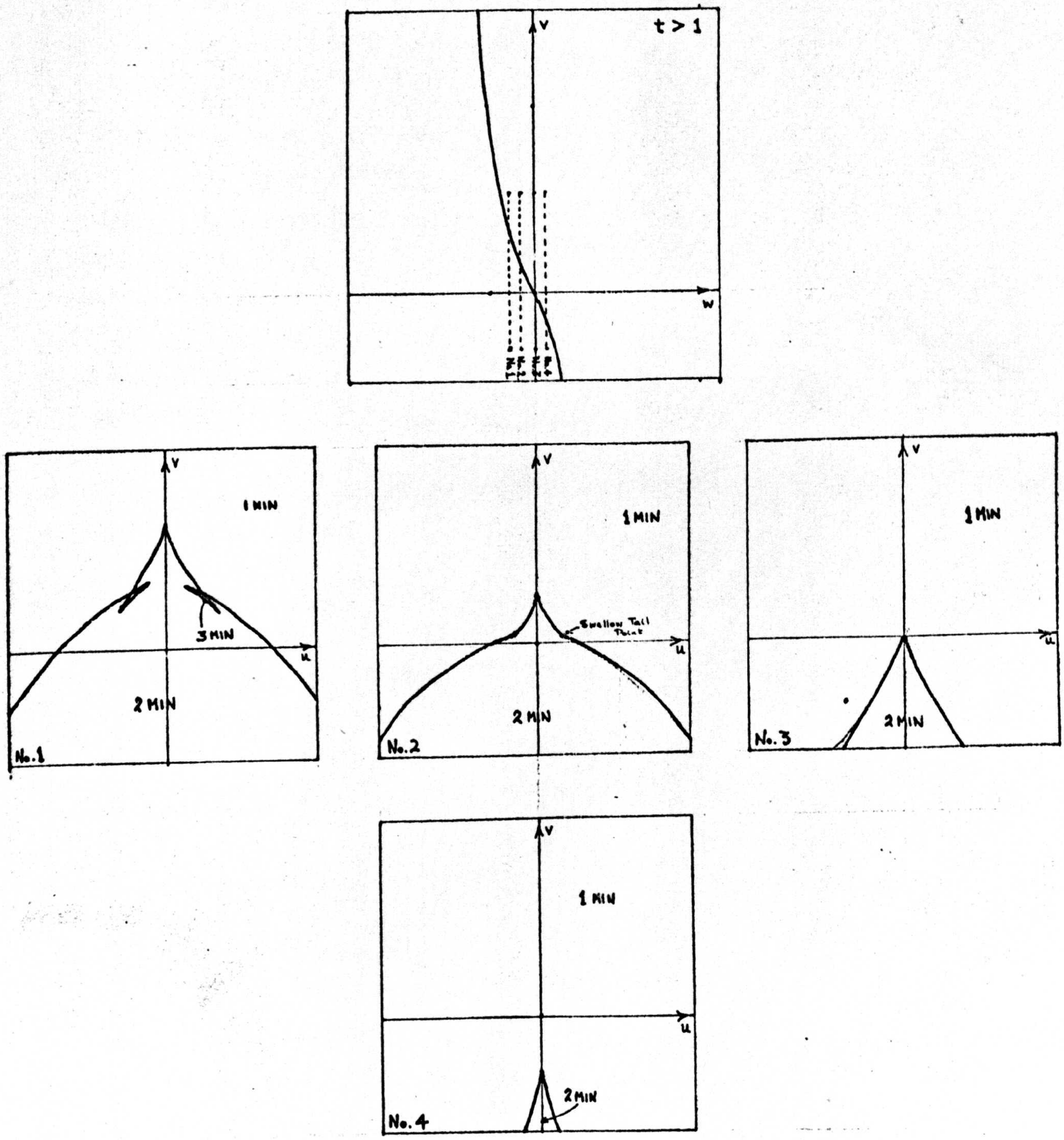


Fig. 12.

values for v used in Fig. 1 and elsewhere. On this we have included dashed lines parallel to the v-axis corresponding to the parts of the v-axis for the $w = \text{const.}$, (u,v)-sections given in 6b. Not all the diagrams of 6b have been given the same scale due to the relative scale of the features of interest. Also not all the different types of (u,v)-graph have been given. Those that have not been given can be deduced fairly easily from those given in Fig. 6 and elsewhere in Figs. 7-12. We have not included the $t \leqslant -\frac{1}{3}$ family since this gives only changes relating to the result of Lemma 12A and the corresponding (u,v)-graph family can easily be constructed.

Figs. 7-12 are constructed on the same pattern as Fig. 6, the (w,v) graph including indications of the w-value and v-range of the corresponding (u,v)-pictures. The transition stages $t = 5/54$, $t = 2/3$ have not been included. The $5/54$ case has not been drawn since it will appear very much the same, as although diffeomorphically distinct from, the general $5/54 < t < 2/3$ case (Fig. 9). This can be seen from the pictures of Fig. 5. For $t = 2/3$ we get the 'beak to beak' point of No. 4 of Fig. 9 at infinity for $w = 0$, and the larger u value '2 MIN' regions of Nos. 5-6 disappear.

§ 6 Conclusions

It has been made clear the the rest of this chapter what results have been obtained and which of these are established analytically and which are purely numerical results. To tidy up this lack of analytic support would be one useful extension of this chapter.

As stated in the introduction to this chapter, the diagrams relate to the unfolding of $x^4 + y^4$. It is possible to extend the given work on a 4-dimensional section of this unfolding to a 5-dimensional section by considering

$$\frac{x^4 + y^4}{4} + \alpha x^2 y + ty^2 + wx^2 - vy - ux \,.$$

It turns out that much of the work of this chapter can be generalised to this new function. The main points to notice are that the straight lines in the $u = 0$, $t = \text{const.}$ sections of Fig. 6-16 have gradient -2α and the line through the origin of Fig. 1-2 is $y = -w/\alpha$. It is not difficult to see that generally $\alpha > 0$ will give very much the same u,v-pictures as our particular $\alpha = 1$. For $\alpha = 0$ and $\alpha < 0$ the details have not been worked out. The programs used for the calculations were in many cases sufficiently general to be used in these situations and the cusp analysis of §4 carries over to general $\alpha \neq 0$. Since much of the complexity of the $x^4 + y^4$ unfolding appears to be tied up with the xy^2 and x^2y terms it may well be that the diagrams of this chapter, together with the $\alpha = 0$, $\alpha < 0$ results will make it possible to understand the unfolding of $x^4 + y^4$.

If we consider just the neighbourhood of the origin in the (u,v,w,t) space and the (x,y) 'state'-space, then we have the same pictures as those of Appendix I in the neighbourhood of the origin. If we allow non local points in the (x,y) space we get a version of the parabolic umbilic without the disadvantage of having the obvious least 'minima' at infinity. To get the pictures we just consider those of Figs. 6-8 in the neighbourhood of $w = u = v = 0$.

References

[1] Thom, R. Stabilité Structurelle et Morphogenese Benjamin (to appear).

[2] Zeeman, E.C. Private conversation.

CHAPTER IV

THE DOUBLE RIEMANN-HUGONIOT CATASTROPHE

In this chapter we give a preliminary survey of the unfolding for the function $V(x,y) \equiv x^4 + y^4$.

Definition 0 A universal unfolding of the function

$$V : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto x^4 + y^4$$

we call a double Reimann-Hugoniot catastrophe.

The reasons for the name will be given in §3 of this chapter.

§1 Concerning Mather's Unfolding

Under C^∞- equivalence we have the Mather unfolding of the germ at the origin of the function $x^4 + y^4$ given by the formula

$$V_m \equiv x^4 + y^4 + \gamma x^2y^2 + \alpha x^2y + \beta xy^2 + wx^2 + \delta xy + ty^2 - ux - vy$$

where $m = (\gamma, \alpha, \beta, \delta, w, t, u, v) \in \mathbb{R}^8$.

Lemma 1 The germ at the origin of the function $V_{m*} \equiv x^4 + y^4 + g\,x^2y^2$ has the unfolding given by

$$V_{m*} + \gamma x^2y^2 + \alpha x^2y + \beta xy^2 + wx^2 + \delta xy + ty^2 - ux - vy$$

for $|g| \neq 2$.

Proof Using Wall's exposition [1]; for the Mather unfolding we find that we need to calculate the ideal I generated by the polynomials V_{m*}, $4x^3 + 2gxy^2$, $4y^3 + 2gx^2y$. Since V_{m*} is homogeneous Euler's theorem tells us that we only need to consider the generators

$$\frac{\partial V_{m*}}{\partial x}, \frac{\partial V_{m*}}{\partial y} .$$

Thus every element of I is of the form

$$P_1(x,y)\, x\, (x^2 + \tfrac{g}{2} y^2) + P_2(x,y)\, y\, (y^2 + \tfrac{g}{2} x^2)$$

for P_1, $P_2 \in R[x,y]$.

With $P_1 = y$, $P_2 = -gx/2$ we get $x^3y(1 - g^2/4) \in I$ and with $P_1 = -gy/2$, $P_2 = x$ we get $xy^3(1 - g^2/4) \in I$. So for $|g| \neq 2$ we can generate x^3y, xy^3 terms and their multiples. Further since

$$xV_{m*} = x^5 + gy\,(x^3y) + y(xy^3)$$

we get $x^5 \in I$ and similarly $y^5 \in I$. Thus all 5th order terms and higher $\subset I$. So the quotient $R[x,y]/I$ has only polynomials with terms in $x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^2y^2, y^4$. Also there are the relationships

$$x^4 = -(g/2)x^2y^2 = y^4 \;;$$

$$x^3 = -(g/2)xy^2 \;; \quad y^3 = (-g/2)x^2y$$

so that, as a linear space over $\mathbb{R}$, the quotient as a ring $R[x,y]/I$ gives the required unfolding. □

<u>Lemma 2</u> The germs of the functions $V_{m*} = x^4 \pm 2x^2y^2 + y^4$ at the origin have an infinite number of C^0-types (a fortiori C^∞-types) in any neighbourhood.

<u>Proof</u> $V_{m*} = (x^2 \pm y^2)^2$ so that as shown by Kuo [2,p235] the jets defined by these polynomials are not C^0 sufficient and hence [2,p227] we get the infinite number of topological types in the neighbourhood, □

So for $|g| = 2$ we find that we could not have expected to get any finite unfolding at all in Lemma 1. We see then that the unfolding is in some sense invarient with respect to the variation of γ for $|\gamma|<2$. The presence of the x^2y^2 term in the unfolding shows, however, that variation of γ will alter the C^∞-type of the germ at the origin anywhere in the range $|\gamma|<2$. However, the topological type is not necessarily altered by changes in γ.

<u>Lemma 3</u> The functions $x^4 + y^4$, $x^4 + \gamma x^2y^2 + y^4 : \mathbb{R}^2 \to \mathbb{R}$ are topological equivalent for $|\gamma| < 2$.

<u>Proof</u> To prove the equivalence we construct a homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$ to satisfy the definition of topological equivalence. Let the coordinates be (x,y), (ξ,η) in the two planes so that the two functions are $x^4 + \gamma x^2y^2 + y^4$ and $\xi^4 + \eta^4$. If we now put $x = r\cos\theta$, $y = r\sin\theta$ and $\xi = \rho\cos\phi$, $\eta = \rho\sin\phi$ then the map

$$\theta = \phi$$

$$r = \rho\sqrt[4]{\frac{\cos^4\phi + \sin^4\phi}{\cos^4\phi + \gamma\cos^2\phi\sin^2\phi + \sin^4\phi}}$$

will be the required homeomorphism for $|\gamma| < 2$. The condition on γ is sufficient to ensure that $x^4 + \gamma x^2y^2 + y^4$ is positive definite. Thus term under the root sign is positive for all ϕ and the function well defined and continuous.

<u>Note</u> (i) The equivalence of the functions includes the C^0-equivalence of their germs at the origin.

(ii) The equivalence does not necessarily mean that for values of α, β, δ, w, t, u, v not all zero, changes in γ do not now alter the topological type of either the germ or the global function.

The unfolding can be described in terms of the infinite dimensional space of germs of C^∞ functions $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. In terms of this description, the results of this section are as follows :

(i) $x^4 + y^4 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ lies a strata C^∞ equivalent germs with codimension 8. (Mather unfolding).

(ii) Putting coordinates transverse to this strata we find along one coordinate direction other germs belonging to different strata of codimension 8 and two germs in a strata of infinite codimension (Lemmas 1 and 2).

(iii) Along the axis of coordinates of the coordinate selected in (ii) we find that we do not alter the topological type of the germs. This topological equivalence spreads along the different C equivalence strata so that $x^4 + y^4$ appears to ~~lies~~ lie in a topological strata of codimension ≤ 7. Clearly much more remains for the complete description of the relationship between the C^∞ and C^0 strata in the neighbourhood of the germ of $x^4 + y^4$.

§ 2 The Local Bifurcation Set

In the usual pictures of the bifurcation set for applications we are concerned with the global behaviour of a family of potential functions. The equations for these potential functions are obtained from consideration of germs, but once derived the local nature is neglected. If we restrict ourselves to a neighbourhood of the origin when looking at the potential functions, we find a rather different bifurcation set. We shall call such a set a <u>local bifurcation set</u>. Clearly there could be a different set for the two cases of topological and smooth equivalence of germs.

To get the local topological bifurcation set for x + y we must consider in detail the topological equivalence of the jets defined by the Mather unfolding. As a first step we prove the following theorem relating to the degrees of sufficiency [3] of the jets in $J^r(2,1)$ given by the Mather unfolding.

<u>Theorem 1</u> The polynomial

$$P(x,y) \equiv x^4 + y^4 + \gamma x^2y^2 + \alpha x^2y + \beta xy^2 + wx^2 + \delta xy + ty^2 - ux - vy$$

represents a jet at the origin that is topologically finitely determined if $|\gamma| < 2$.

<u>Proof</u>. We prove this theorem by establishing the degree of sufficiency for the jet in all cases. These results require the use of two methods

proved by Kuo, the first [4] involving projective varieties

$$H_1 \equiv ux + vy \quad ;$$
$$H_2 \equiv wx^2 + \delta xy + ty^2 \quad ;$$
$$H_3 \equiv \alpha x^2 y + \beta xy^2 \quad ;$$
$$H_4 \equiv x^4 + \gamma x^2 y^2 + y^4 \quad ;$$

and the second [2] requiring local factorisation of $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$. For u, v not both zero H_1 determines the topological type of the jet so that we have a C^0 - 1 - determined jet. The degree of sufficiency is thus 1. For $u = v = 0$ but $\delta^2 \neq 4wt$ we have $H_2 = 0$ a non-singular projective variety so that by the first method [4] the jet has degree of sufficiency 2.

For $u = v = 0$ and $\delta^2 = 4wt$ we can subdivide into five cases (A) $H_2 \equiv 0$ i.e. $\delta = w = t = 0$; (B) $\delta = w = 0$, $t \neq 0$, (C) $\delta = t = 0$, $w \neq 0$; (D) $wt \neq 0$, Kuo's first method applies; (E) $wt \neq 0$, Kuo's first method does not give a result.

We begin with the most difficult cases, $wt \neq 0$ and see when the sufficient conditions given by Kuo's first method apply. The variety $H_2 = 0$ has a repeated factor and if $H_3 = 0$ has distinct factors none of which are common to $H_2 = 0$ then the degree of sufficiency of the jet is 3. In terms of parameters this corresponds to $\alpha\beta \neq 0$, $\alpha/\beta \neq \pm\sqrt{w/t}$, (+ if $\delta > 0$, - if $\delta < 0$), $wt \neq 0$, $\delta^2 = 4wt$, $u = v = 0$.

When H_3 is not trivial and $H_3 = 0$ has a repeated factor, i.e. $\alpha = 0$, $\beta \neq 0$ or $\alpha \neq 0$, $\beta = 0$, then H_3 has no factor in common with $H_2 (wt \neq 0)$. For $|\gamma| < 2$, the form H_4 has no real factors so that it has no common factors with either H_2 or H_3. The degree of sufficiency is thus 4 when $\alpha\beta = 0$, $\alpha + \beta \neq 0$, $wt \neq 0$, $\delta^2 = 4wt$, $u = v = 0$. With $\alpha = \beta = 0$ and $H_3 \equiv 0$ we find that the degree of sufficiency is still 4.

The last two paragraphs have dealt with (D) so that we have for the remaining $wt \neq 0$ the case of jets of the form

$$k(\alpha x + \beta y)^2 + xy(\alpha x + \beta y) + x^4 + \gamma x^2 y^2 + y^4$$

with $w = k\alpha^2$, $t = k\beta^2$, $\delta = 2k\alpha\beta$, i.e. $\delta^2 = 4wt$, $\alpha\beta \neq 0$. To deal with these we can use Kuo's second method. To apply this we first use the transformation (a rotation and magnification)

$$X = \alpha x + \beta y$$
$$Y = -\beta x + \alpha y$$

and get

$$F(x,y) = kX^2 + \alpha\beta X^3/(\alpha^2+\beta^2) + (\alpha^2-\beta^2)X^2Y/(\alpha^2+\beta^2)^2$$
$$+ ((\alpha^4+\gamma\alpha^2\beta^2+\beta^4)X^4 + 2\alpha\beta(\alpha^2-\beta^2)(\gamma-2)X^3Y$$
$$+ (12\alpha^2\beta^2+\gamma(\alpha^4-4\alpha^2\beta^2+\beta^4))X^2Y^2 + 2\alpha\beta(\alpha^2-\beta^2)(2-\gamma)XY^3$$
$$+ (\alpha^4+\gamma\alpha^2\beta^2+\beta^4)Y^4)/(\alpha^2+\beta^2)^4$$

with partial derivatives

$$\partial F/\partial X = 2kX + 3\alpha\beta X^2/(\alpha^2+\beta^2)^2 + 2(\alpha^2-\beta^2)XY/(\alpha^2+\beta^2)^2$$
$$- \alpha\beta Y^2/(\alpha^2+\beta^2)^2 + 4(\alpha^4+\gamma\alpha^2\beta^2+\beta^4)X^3/(\alpha^2+\beta^2)^4$$
$$+ 6\alpha\beta(\alpha^2-\beta^2)(\gamma-2)X^2Y/(\alpha^2+\beta^2)^4 + 2(12\alpha^2\beta^2+\gamma(\alpha^4-4\alpha^2\beta^2+\beta^4))XY^2/(\alpha^2+\beta^2)^4$$
$$+ 2\alpha\beta(\alpha^2-\beta^2)(2-\gamma)Y^3/(\alpha^2+\beta^2)^4$$

$$\partial F/\partial Y = (\alpha^2-\beta^2)X^2/(\alpha^2+\beta^2) - 2\alpha\beta XY/(\alpha^2+\beta^2)^2$$
$$+ 2\alpha\beta(\alpha^2-\beta^2)(\gamma-2)X^3/(\alpha^2+\beta^2)^4 + 2(12\alpha^2\beta^2+\gamma(\alpha^4-4\alpha^2\beta^2+\beta^4))X^2Y/(\alpha^2+\beta^2)^4$$
$$+ 6\alpha\beta(\alpha^2-\beta^2)(2-\gamma)XY^2/(\alpha^2+\beta^2)^4 + 4(\alpha^4+\gamma\alpha^2\beta^2+\beta^4)Y^3/(\alpha^2+\beta^2)^4.$$

In the general case it is useful to be able to assume that $\neq$ so that we deal with the special case = first. In this case we have

$$\partial F/\partial X = 2kX \pm (3/4\alpha^2)X^2 \mp (1/4\alpha^2)Y^2 + (2+\gamma)X^3/8\alpha^4$$
$$+ (6-\gamma)XY^2/4\alpha^4 \;;$$
$$\partial F/\partial Y = \mp(1/2\alpha^2)XY + (6-\gamma)X^2Y/4\alpha^4 + (2+\gamma)Y^3/4\alpha^4 \;;$$

with positive sign for $\alpha = \beta$ and the negative sign with $\alpha = -\beta$. To apply Kuo's method we need to calculate the factors of $\partial F/\partial X$ and $\partial F/\partial Y$ of the form $(X - \bar{Y})$ where $\bar{Y}$ is a fractional power series in Y of order > 1. For $\partial F/\partial X$ we get as a third approximation for the only relevant factor (using method given by Walker [5, pp.97-106]),

$$X - Y^2(\pm(1/8k\alpha^2) + Y^2(\pm(1/64k\alpha^6)(\gamma - (6 + 3/8k)) +$$
$$+ Y^2(\pm(1/512k^2\alpha^{10})(\gamma^2 - (12 + 9/8k + 1/8k^2)\gamma +$$
$$(36 + 27/8k + 1/32k^2)) + X_3)))$$

if $\gamma \neq 6 + 3/8k$ and $\gamma^2 - (12 + 9/8k + 1/8k^2)\gamma + (36 + 27/8k + 1/32k^2) \neq 0$ for $|\gamma| < 2$. If $\gamma^2 - (12 + 9/8k + 1/8k^2)\gamma + (36 + 27/8k + 1/32k^2) = 0$ for some relevant γ then we must use, in places of the vanishing term, the approximation

$$Y(\pm 3(\gamma - 6 - 3/8k)((2 + \gamma)/k + (\gamma - 6 - 3/8k)/64)/$$
$$512k^3\alpha^{14} + X_3).$$

In case $\gamma = 6 + 3/8k$, $|\gamma| < 2$ we get the factor (to second order approximation)

$$X - Y^2(\pm 1/8k\alpha^2 + Y^4(\mp(2 + \gamma)/4096k^3\alpha^{10} + X_2)).$$

If we now substitute these as expressions for X in $\partial F/\partial Y$ we can calculate what Kuo denotes as n_1 [2, p.227]. We do not need to back substitute into $\partial F/\partial X$ since the factors are real to sufficiently high order of approximation. The results after a considerable amount of elementary algebra are as follows:

(a) $\gamma = 6 + 3/8k$, k such that $|\gamma| < 2$, $-3/32 < k < -3/64$; $n_1 = 3$.

(b) $\gamma \neq 6 + 3/8k$, $\gamma \neq -2 + 1/4k$; $n_1 = 3$.

(c) $\gamma \neq 6 + 3/8k$, $\gamma = -2 + 1/4k$, $|\gamma| < 2 \Rightarrow k > 1/16$ and since $16k^2 + 17k/2 - 1/4 \neq 0$ for such k we get $n_1 = 4$.

Using the same methods we can get the relevant factors for F/ Y as

$$X - Y^2(\pm(2 + \gamma)/4\alpha^4 + Y^2(\pm(6 - \gamma)(2 + \gamma)^2/8\alpha^6 + Y^2(\pm(6 - \gamma)^2(2 + \gamma)^3$$
$$/16\alpha^{10} + X_3)))$$

all $|\gamma| < 2$ and substitution in $\partial F/\partial X$ gives values for m_1 as follows :

(a) $\gamma \neq -2 + 1/4k$, $m_1 = 2$.

(b) $\gamma = -2 + 1/4k$, $k \neq -1/8$ $k > 1/16$ (i.e. $|\gamma| < 2$), $m_1 = 4$.

(c) $\gamma = 0$, $k = 1/8$, $m_1 = 6$.

Combining the results for m_1, n_1 we get the degree of sufficiency for the jet -

(a) $\gamma \neq -2 + 1/4k$, degree of sufficiency = 4;

(b) $\gamma = -2 + 1/4k \neq 0$, i.e. $k > 1/16$, $k \neq 1/8$ degree of sufficiency = 5;

(c) $\gamma = 0$, $k = 1/8$, degree of sufficiency = 7.

Returning to the case $\alpha \neq \pm\beta$ we must factorise the general expressions (1), (2). For $\partial F/\partial X$ we get

$$X - Y^2(\alpha\beta/2k(\alpha^2+\beta^2)^2 + Y(-\alpha\beta(\alpha^2-\beta^2)(2-\gamma+1/2k)/k(\alpha^2+\beta^2)^4 + X_2)),$$

if $\gamma \neq 2 + 1/2k$. In the special cases we have $\gamma = 2 + 1/2k$ and get factors as follows :

(i) $X - Y^2(\alpha\beta/2k(\alpha^2+\beta^2)^2 + Y^2(-\alpha\beta(4k+1)(\alpha^4+(16k-5)\alpha^2\beta^2/$
$2(4k+1)+\beta^4)+X_2))$
with $k = 1/4$ not being a special case except for vanishing of α^4, β^4 terms in the innermost bracket.

(ii) There are real values of α, β giving

$$\alpha^4 + \frac{16k-5}{2(4k+1)}\alpha^2\beta^2 + \beta^4 = 0$$

when $-1/4 < k \leqslant 1/32$ ($k = 1/32$ gives $\alpha = \pm\beta$) and for $k < -1/8$ we satisfy the condition $|\gamma| < 2$. Thus in some cases we need the approximation

$$X - Y^2(\alpha\beta/2k(\alpha^2+\beta^2)^2 + Y^3(-3\alpha^3\beta^3(\alpha^2-\beta^2)/8k^4(\alpha^2+\beta^2)^8 + X_2)) .$$

Substituting these expressions for X in (2) we get the following values for n_1 -

(a) $\gamma \neq 2 + 1/2k$, $(\alpha/\beta)^4 + (\gamma - 1/4k)(\alpha/\beta)^2 + 1 \neq 0$, $n_1 = 3$;

(b) $\gamma \neq 2 + 1/2k$, $(\alpha/\beta)^4 + (\gamma - 1/4k)(\alpha/\beta)^2 + 1 = 0$, which can happen for $k > 1/16$, $n_1 = 4$;

(c) $\gamma = 2 + 1/2k$, $n_1 = 4$.

Similarly for the relevant factors of (2) we get

$$X - Y^2(2(\alpha^4 + \gamma\alpha^2\beta^2 + \beta^4)/\alpha\beta(\alpha^2+\beta^2)^2 + Y(2(\alpha^2-\beta^2)(\alpha^4+\gamma\alpha^2\beta^2+\beta^4)$$
$$(3(2-\gamma)/(\alpha^2+\beta^2)^2 + (\alpha^4+\gamma\alpha^2\beta^2+\beta^4)/\alpha^2\beta^2)/\alpha\beta(\alpha^2+\beta^2)^4 + X_2)),$$

for all $\alpha^2 \neq \beta^2$, $|\gamma| < 2$. The substitution then gives -

(a) $k > 1/16$, $|\gamma| < 2$, $(\alpha/\beta)^4 + (\gamma - 1/4k)(\alpha/\beta)^2 + 1 = 0$, $m_1 = 3$;

(b) $m_1 = 2$ otherwise.

Thus using m_1 and n_1 we get the degrees of sufficiency -

(a) $\gamma \neq 2 + 1/2k$, $(\alpha/\beta)^4 + (\gamma - 1/4k)(\alpha/\beta)^2 + 1 \neq 0$, degree of sufficiency = 4;

(b) otherwise, degree of sufficiency = 5.

This completes the cases (D), (E) in the list of subcases for $\delta^2 = 4wt$.

The two cases (B), (C) are symmetric under the pair of changes $w \leftrightarrow t$, $\alpha \leftrightarrow \beta$, so that we need only consider one of them in detail. For (C) we then get the jet

$$Z(x,y) = ty^2 + \alpha x^2 y + \beta y^2 x + x^4 + \gamma x^2 y^2 + y^4$$

with

$$Z_x(x,y) = 2\alpha xy + \beta y^2 + 4x^3 + 2\gamma x y^2 \ ;$$

$$Z_y(x\ y) = 2ty + \alpha x^2 + 2\beta xy + 2\gamma x^2 y^2 + 4y^3.$$

If $\alpha = \beta = 0$ then the first method applies and degree of sufficiency is 4. If $\alpha = 0$ then we can assume $\beta \neq 0$ and get by elementary factorisation, Z_x with factors ,

$$y - \left(\pm \frac{2}{\sqrt{-\beta}} \ x^{\frac{3}{2}} \left(1 + \frac{2\gamma}{\beta} x \right)^{-\frac{1}{2}} \right)$$

and Z_y with factors

$$y \ , \ y - \left(\pm \left(\frac{t + \beta x + \gamma x^2}{2} \right)^{\frac{1}{2}} \right).$$

For $\beta < 0$ we get $m_1 = 3$, $n_1 = 3/2$ so that degree of sufficiency is 4. With $\beta > 0$ we get only $y = 0$ to substitute in Z_y, since real part of Z_x factor vanishes, this gives $n_1 = 0$ and we still get 4 as the degree of sufficiency.

If in (C) we have $\alpha \neq 0$ then this produces three subcases $\beta \neq 0$; $\beta = 0, \gamma \neq 0$; $\beta = \gamma = 0$. The corresponding factors of Z_x for Kuo's method are :

$$y - x^2 \left(-\frac{2}{\alpha} + x \left(\frac{-2\beta}{\alpha^3} + y_2 \right) \right) \qquad \alpha \neq 0 \ , \ \beta \neq 0 \ ;$$

$$y - x^2 \left(-\frac{2}{\alpha} + x^2 \left(-\frac{4\gamma}{\alpha^3} + y_2 \right) \right) \quad \beta = 0, \ \alpha \neq 0, \ \gamma \neq 0 \ ;$$

$$y + \frac{2x^2}{\alpha} \qquad \alpha \neq 0 \ , \ \beta = \gamma = 0 \ .$$

Substitution in Z_y gives

(a) $\alpha^2 \neq 4t$; $\alpha \neq 0$; $n_1 = 2$,

(b) $\alpha^2 = 4t$; $\alpha \neq 0$; $\beta \neq 0$ or $\beta = 0$ and $\gamma \neq 0$; $n_1 = 3$,

(c) $\alpha^2 = 4t$; $\alpha \neq 0$; $\beta = \gamma = 0$; $n_1 = 6$.

Factorising Z_y we have in the same cases

$$y - x^2 \left(-\frac{\alpha}{2t} + x \left(\frac{\beta\alpha}{2t^2} + x \left(\frac{\alpha}{2t^2} \left(\frac{\beta^2}{t} - \gamma \right) + y_3 \right) \right) \right) \qquad \alpha\beta \neq 0 \ ;$$

(the case $\gamma = \beta^2/t$ is not relevent to substitutions);

$$y - x^2 \left(-\frac{\alpha}{2t} + x^2 \left(\frac{\gamma\alpha}{2t^2} + y_2 \right) \right) \quad \alpha \neq 0 \ , \ \beta = 0 \ , \ \gamma \neq 0 \ ;$$

$$y = x\left(-\frac{\alpha}{2t} + x^4\left(\frac{\alpha^3}{4t^4} + y_2\right)\right) \qquad \alpha \neq 0\,,\ \beta = \gamma = 0\ ;$$

with substitution in Z_x giving

(a) $\alpha^2 \neq 4t$; $\alpha \neq 0$; $m_1 = 3$;

(b) $\alpha\beta \neq 0$, $\alpha^2 = 4t$, $\alpha \neq -\beta/4$; $m_1 = 4$;

(c) $\alpha\beta \neq 0$; $\alpha^2 = 4t$; $\alpha = -\beta/4$ (the case $\gamma = \beta^2/t$ which might be relevant gives $\gamma = 64 > 2$); $m_1 = 5$;

(d) $\alpha \neq 0$; $\beta = 0$; $\alpha^2 = 4t$; $m_1 = 5$.

From these values of m_1 , n_1 we get as degree of sufficiency cases

(a) $\alpha \neq 0$; $\alpha^2 \neq 4t$; degree of sufficiency = 4 ;

(b) $\alpha\beta \neq 0$; $\alpha^2 = 4t$; $\alpha \neq -\beta/4$; degree of sufficiency = 5 ;

(c) $\alpha\beta \neq 0$; $\alpha^2 = 4t$; $\alpha = -\beta/4$, $|\gamma| < 2$ or $\alpha \neq 0$; $\beta = 0$
$\gamma \neq 0$, $\alpha^2 = 4t$; degree of sufficiency = 6 ;

(d) $\alpha \neq 0$, $\beta = \gamma = 0$, $\alpha^2 = 4t$; degree of sufficiency = 7 ;

and this completes the cases (B) - (E) for $\delta^2 = 4wt$.

To deal with (A) we have the polynomial

$$xy\,(\alpha x + \beta y) + x^4 + \gamma x^2 y^2 + y^4$$

which is clearly C^0 - 3 - determined if $\alpha\beta \neq 0$. By Lu's result [3] on the parabolic umbilic the jet is C^0 - 4 determined for $\alpha = 0$, $\beta \neq 0$ and $\alpha \neq 0$, $\beta = 0$. Both these latter conclusions hold for all . We have the final cases of $u = v = w = \delta = t = \alpha = \beta = 0$ only H_4 remaining non-trivial and this jet has degree of sufficiency 4 for $|\gamma| \neq 2$, a fortiori for $|\gamma| < 2$. $\square$

The restriction $|\gamma| < 2$ is useful in reducing the working at several stages in the calculation of the degree of sufficiency and clearly for $\gamma = \pm 2$ we can get some choices of the other parameters for which the jet is not C^0-sufficient.

<u>Examples</u> (i) $x^4 - 2x^2y^2 + y^4$ is not C^0-sufficient;

(ii) $x^4 - 2x^2y^2 + y^4 + x^2 + 2xy + y^2$ is not C^0-sufficient

since we can write as

$$(x + y)^2 + (x + y)\left[(x^2 - y^2)(x - y)\right]$$

and complete the square.

This last example suggests that there might be other similar cases.

<u>Lemma 4</u> For $\gamma < -2$ there are choices of the other parameters for which the jet $P(x\ y)$ of Theorem 1 is not C^0-sufficient.

<u>Proof</u> If $\gamma<-2$ there are real quadratic factors

$$\left(x^2+\left(\frac{\gamma+\sqrt{(\gamma^2-4)}}{2}\right)y^2\right)\left(x^2+\left(\frac{\gamma-\sqrt{\gamma^2-4}}{2}\right)y^2\right)$$

for H_4 which can be factorised again into real linear factors. Let $(x+p_1y)$ be one of these linear factors and the jet

$$(x+p_1y)^2+(x+p_1y)\left[(x-p_1y)(x+p_2y)(x-p_2y)\right]$$

is clearly not C^0-sufficient. □

(For $\gamma>2$ the proof of this Lemma does not hold since we do not have real linear factors for H_4.)

In the parameter space, $\mathbb{R}^8$ for the polynomial $P(x,y)$ of Theorem I we can define equivalence classes by degree of sufficiency. We have in Theorem I described the algebraic geometry of these equivalence classes in an 8-dimensional region containing the origin, but the complete description of the geometry is still lacking.

<u>Corollary 1 to Theorem 1</u> The Polynomial $P(x\ y)$ of Theorem 1 is v-sufficient to some order if $|\gamma|<2$.

<u>Proof</u> C^0-sufficient implies v-sufficient [2, p.226] □

<u>Colloary 2 to Theorem 1</u> If $P(x\ y)$ is as in Theorem 1, then the algebraic variety $P(x\ y)=0$ has an isolated singularity at $(0,0)$ if $|\gamma|<2$.

<u>Proof</u> If $\partial P/\partial x$, $\partial P/\partial y$ have a common factor in a neighbourhood of the origin P is not C^0-sufficient to any order [2, p.234-235]. But by Theorem 1 $P(x\ y)$ is C^0-sufficient. Thus $\partial P/\partial x$, $\partial P/\partial y$ have no common factors. Hence grad $P(x,y)\neq 0$ except for $x=y=0$ in some neighbourhood of the origin. □

Thom has claimed and Kuo has questioned [6] the proof of the very reasonable result that the topological type of map $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ is determined by the topological type of the set $f^{-1}(0)$, if grad $f(x)\neq 0$ except at $x=0$. With this result established we can get at the local topological unfolding of x^4+y^4 by using some method to calculate the real factors of $P(x,y)=0$ in the neighbourhood of the origin. This is something the same as the proof given for Theorem 1 and we follow through the cases in the proof of Theorem 1 closely.

<u>Definition</u> The jet $P(x\ y)$ is of <u>type-n</u> if the germ of $P^{-1}(0)$ is topologically equivalent to n distinct lines through the origin $n\geqslant 1$. $P^{-1}(0)$ is <u>type-0</u> if it is an isolated point.

<u>Note</u>: This definition does not claim to be an exhaustive classification of

all germs $P^{-1}(0)$ although it may be so.

Lemma A For $P(x,y)$ of Theorem 1, $P^{-1}(0)$ is of type-1 if u,v not both zero.

Proof The jet is 1-sufficient, so topologically determined by $ux + vy$. Thus $P^{-1}(0) \sim ux + vy = 0$ i.e. of type-1. $\square$

Lemma B For $P(x,y)$ of Theorem 1, $u = v = 0$, $\delta^2 \neq 4wt$ we have type-0 if $\delta^2 < 4wt$ and type-2 if $\delta^2 > 4wt$.

Proof The jet is 2-sufficient so we just consider the quadratic terms. $\square$

Lemma C For $P(x,y)$ of Theorem 1, $u = v = 0$, $\delta = w = t = 0$ then we have ; $\alpha\beta \neq 0$ - type-3 ; $\alpha = 0$, $\beta \neq 0$ and $\alpha \neq 0$, $\beta = 0$ - type-2 ; $\alpha = \beta = 0$ - type-0.

Proof For $|\gamma| < 2$, $\alpha = \beta = 0$ gives $P^{-1}(0)$ as

$$x^4 + \gamma x^2 y^2 + y^4 = 0$$

Clearly the origin only and type-0.

For $\alpha = 0$, $\beta \neq 0$ and $\beta = 0$, $\alpha \neq 0$ we have by Lu's result [3] that this is topologically equivalent to $x^2 y + y^4$ and there $P^{-1}(0)$ is clearly of type-2.

With $\alpha\beta \neq 0$ the jet is 3-sufficient so that we need only consider $H_3 = 0 \equiv \alpha x^2 y + \beta x y^2 = 0$ and $P^{-1}(0)$ is of type-3. $\square$

Lemma D For $P(x,y)$ of Theorem 1, $u = v = 0$, $\delta^2 = 4wt$, $\delta = w = 0$ $t \neq 0$ $\alpha^2 > 4t$ we have type-2.

Proof By Theorem 1 we have degree of sufficiency ≥ 4 for this case and we must consider the variety.

$$P(x,y) \equiv ty^2 + \alpha x^2 y + \beta x y^2 + x^4 + \gamma x^2 y^2 + y^4 = 0 \text{ to find } P^{-1}(0).$$

By the result given by Walker [5 p.106, Th. 3.2] there is a unique factorisation

$$(y - \bar{y}^{(1)})\ (y - y^{(2)})\ (y - y^{(3)})\ (y - \bar{y}^{(4)})$$

where $\bar{y}^{(i)}$ are complex fractional power series. Further we know that the orders of $\bar{y}^{(i)}$ are all non-negative [5 p.106, Th. 3.3].

Drawing the Newton polygon and continuing the method of factorisation we find two factors of zero order. These factors are of no interest to the determination of the germ of $P^{-1}(0)$ in the neighbourhood of the origin. The Newton's polygon method however, also gives two roots

$$y = c_1 x^2 + a_3 x^3 + \ldots$$

where c_1 is a real root of $tc_1^2 + \alpha c_1 + 1 = 0$ $(\alpha^2 > 4t)$, and $a_3, \ldots$ are real numbers. Thus $P^{-1}(0)$ has two real curves through the origin and is of type-2. $\square$

Lemma E For $P(x,y)$ of Theorem 1 $u = v = 0$, $\delta = w = 0$, $t \neq 0$, $\alpha^2 \neq 4t$, the germ of $P^{-1}(0)$ is of type-0.

Proof As for Lemma D we get the Newton polygon with the only relevant segment joining (0,4), (1,2), (2,0) and we use the substitution $y = x^2(c_1 + y_1)$ to give

$$x^4(1 + c_1\alpha + tc_1^2) + c_1^2\beta x^5 + c_1^2\gamma x^6 + c_1^4 x^8 +$$
$$y_1\left[(\alpha + 2tc_1)x^4 + 2\beta c_1 x^5 + 2\gamma c_1 x^6 + 4c_1^3 x^8\right] +$$
$$y_1^2\left[tx^4 + \beta x^5 + \gamma x^6 + 6c_1^2 x^8\right] + y_1^3(4c_1 x^8) + y_1^4 x^8$$

We then choose c_1 as a complex root of $1 + c_1\alpha + tc_1^2 = 0$ and note that $(\alpha + 2tc_1) \neq 0$ for $\alpha^2 \neq 4t$. We are thus reduced to the linear equation situation for the determination of all the further coefficients [5, p.101-102]. With c_1 complex and the further coefficients a_3 in the extension field $\mathbb{R}[c_1]$ we have only $x = y = 0$ satisfying the equation

$$y = c_1 x^2 + a_3 x^3 + \ldots$$

Thus we have $P^{-1}(0)$ of type-0. $\square$

Note: The assumption of non-zero radius of convergence for the power series is sufficient to prove that $x = y = 0$ is only real point .

Lemma F For $P(x,y)$ of Theorem 1, $u = v = 0$, $\delta = w = 0$, $t \neq 0$, if $\alpha^2 = 4t$ then we get; $\beta \neq 0$ - type-1; $\beta = 0, \gamma > 0$ - type-2 ; $\beta = 0, \gamma < 0$ - type-0.

Proof Working as in Lemmas D, E we get $y = x^2(c_1 + y_1)$ with $c_1 = -\alpha/2t = -2/\alpha$, and this gives remainder

$$c_1^2\beta x^5 + c_1^2\gamma x^6 + c_1^4 x^8$$
$$+ y_1(2\beta c_1 x^5 + 2\gamma c_1 x^6 + 4c_1^3 x^8)$$
$$+ y_1^2(tx^4 + \beta x^5 + \gamma x^6 + 6c_1^2 x^8) + y_1^3(4c_1 x^8)$$
$$+ y_1^4 x^8,$$

The Newton polygon method gives substitutions

$$y_1 = x^{\frac{1}{2}}(c_2 + y_2) \qquad \beta \neq 0 ;$$
$$y_1 = x(c_2 + y_2) \qquad \beta = 0, \gamma \neq 0 ;$$
$$y_1 = x^2(c_2 + y_2) \qquad \beta = \gamma = 0$$

which leads to the equations

$$c_2^2 = -16\beta/\alpha^4 \qquad \beta \neq 0\,;$$

$$c_2^2 = -16\gamma/\alpha^4 \qquad \beta = 0,\ \gamma \neq 0\,;$$

$$c_2^2 = -64/\alpha^4 \qquad \beta = \gamma = 0\,.$$

For $\beta<0$ we get a real value for c_2 and two real series in $x^{\frac{1}{2}}$ beginning $-2x^2/\alpha \pm 4\sqrt{-\beta}x^{\frac{5}{2}}/\alpha^2 + \dots$

This gives two locii in the $x \geq 0$ half plane i.e. we get a single cusped curve (cusp at the origin) from the pair of series. Hence for $\beta<0$ we have $P^{-1}(0)$ of type-1. If we change the sign of both β and x we get exactly the same situation so that for $\beta>0$ we have a cusped curve in $x \leq 0$ half plane and again type-1. For the $\beta = 0$ cases we have only integer powers of x involved and the methods of Lemmas D and E give the results. □

<u>Lemma G</u> The three cases of $P(x,y)$ with $u = v = 0$, $\delta = t = 0$, $w \neq 0$ given by $\delta^2 > 4w$; $\delta^2 < 4w$; $\delta^2 = 4w$ have topological types analogous to results of Lemmas D, E, F. □

<u>Lemma H</u> If $P(x,y)$ of Theorem 1 has $u = v = 0$, $\delta^2 = 4wt$, $wt \neq 0$ and satisfies the further conditions $\mathrm{sign}(w)\delta>0$, $\alpha/|w|\sqrt{|t|} \neq \beta/|t|\sqrt{|w|}$ then $P^{-1}(0)$ is of type-1.

<u>Proof</u> The Polynomial $P(x,y)$ is of the form

$$\mathrm{sign}(w)\,(\sqrt{|w|}\,x + \sqrt{|t|}\,y)^2 + \alpha x^2 y + \beta x y^2 + x^4 + \gamma x^2 y^2 + y^4$$

and by Theorem 1 is a 3 - sufficient jet. Putting $x = (X - Y)/\sqrt{|w|}$, $y = Y/\sqrt{|t|}$ and ignoring H_4 we get

$$(\alpha/|w|\sqrt{|t|} - \beta/|t|\sqrt{|w|})\,Y^3 + X\left\{(\beta/|t|\sqrt{|w|} - 2\alpha/|w|\sqrt{|t|})\,Y^2\right\}$$
$$+ X^2\left\{\mathrm{sign}(w) + (\alpha/|w|\sqrt{|t|})\,Y\right\}.$$

Using the Newton polygon method we get

$$X = C_1 Y^{\frac{3}{2}} + A_2 Y^2 + \dots$$

where C_1 is given by $C_1^2 = \mathrm{sign}(w)\,(\beta/|w|\sqrt{|t|} - \alpha/|w|\sqrt{|t|})$ and $A_2, \dots$ are given by linear equations. When C_1 is real we get a single cusped curve in $Y \geq 0$ half plane given by two roots for C_1 so that $P^{-1}(0)$ is of type-1. With C_1^2 with the opposite sign we can reduce to the above case by changing the sign of Y. Thus we get type-1 again with the single cusped curve in $Y \leq 0$ half plane. □

<u>Lemma I</u> There is a result corresponding to Lemma H in the $\mathrm{sign}(w)\delta<0$ and $\alpha/|w|\sqrt{|t|} \neq -\beta/|t|\sqrt{|w|}$ case. □

<u>Lemma J</u> $P(x,y)$ of Theorem 1 with $u = v = 0$, $\delta^2 = 4wt$, $wt \neq 0$, $\mathrm{sign}(w)\delta>0$ satisfies the further condition $\alpha/|w|\sqrt{|t|} = \beta/|t|\sqrt{|w|}$.

$P^{-1}(0)$ has type-2 if $\mathrm{sign}(w)\, C_1^2 - (\alpha/|w|\sqrt{|t|})C_1 + (1/w^2 + y/wt + 1/t^2) = 0$ has distinct real roots for C_1 and has type-0 if the quadratic equation has complex roots.

<u>Proof</u> The substitution of Lemma G, this time leaving in the H_+ terms, gives

$$P(x,y) = (1/w^2 + y/wt + 1/t^2)\,Y^5 + X\{(-\alpha/|w|\sqrt{|t|})Y^2 - (4/w^2 + 2y/wt)Y^3\}$$
$$+ X^2\{\mathrm{sign}(w) + (\alpha/|w|\sqrt{|t|})\,Y + (6/w^2 + y/wt)Y^2\}$$
$$+ X^3\{(-4/w^2)Y\} + X^4\{1/w^2\}\,.$$

Using the Newton polygon factorisation we get

$$X = C_1 Y^2 + A_3 Y^3 + \ldots$$

where C_1 is a root of the quadratic equation given in the statement of the Lemma provided this equation does not have a double root. The later coefficients are given by linear equations so that the result depends on the arguments of Lemmas D and E. □

<u>Lemma K</u> $P(x,y)$ of Theorem 1 is as in Lemma J except that the coefficients satisfy the further condition

$$\alpha^2/4w^2t = (1/w^2 + y/wt + 1/t^2)$$

and the quadratic equation of Lemma J has the double root $C_1 = \alpha/2w/\sqrt{|t|}$. For $w \neq t$ we get type-1 for $P^{-1}(0)$.

<u>Proof</u> We put $X = Y^2(C_1 + X_1)$ and get the value for C of the statement of the Lemma from the form of $P(x,y)$ given in Lemma J. Making use of the conditions on the parameters we get

$$\alpha/w/\sqrt{|t|}\ (1/t^2 - 1/w^2)Y^5 + (6/w^2 + y/wt)\,(\alpha^2/4w^2|t|)Y^6 + \ldots$$
$$+ X_1\{(2(2/t + y/w)/t)\ Y^5 + (\alpha(6/w^2 + y/wt)/|w|\sqrt{|t|})Y^6 + \ldots\}$$
$$+ X_1^2\{\mathrm{sign}(w)\ Y^4 + \ldots\} + \ldots$$

The substitution $X_1 = Y^{\frac{1}{2}}(C_2 + X_2)$ gives the quadratic equation

$$C_2^2 = \alpha(1/t^2 - 1/w^2)/|w|\sqrt{|t|}$$

which has non-zero roots if $w \neq t$. This last conclusion follows from (i) $\alpha \neq 0$ is ~~zkz~~ $|y|<2$ since $\alpha^2/4w^2t = (1/w^2 + y/wt + 1/t^2)$; (ii) w,t have the same sign, since $wt = \delta^2/4 > 0$. The result now follows by the arguments used in Lemma F and Lemma H to establish type-1 cases. □

<u>Lemma L</u> $P(x,y)$ of Theorem 1 is as in Lemma K except that $w = t$. For $w<0$ we have $P^{-1}(0)$ of type-2 and for $w>0$ we have type-0. [<u>Note</u>: $w = 0$ is excluded since $wt \neq 0$.]

<u>Proof</u> The Newton polygon method gives C_1 as in Lemma K and after using all the conditions satisfied by the parameters the equation

$$\mathrm{sign}(w)\, C_2^2 + (\alpha^2/2w^3)C_2 + (\alpha^2(4 + \alpha^2/4w)/4|w|w^4) = 0$$

for C_2. This has discriminat $-4\alpha^2/w^5$ which is non zero, since $\alpha \neq 0$ was shown to hold in Lemma K. Since the method will only give integer powers for Y in this case and further coefficients determined by linear equations, we get result by argument given first in Lemma D. □

Lemma M There are results directly analogous to those of J, K, L for $\mathrm{sign}(w).\delta < 0$ and $\alpha/|w|\sqrt{|t|} = -\beta/|t|/\sqrt{|w|}$. □

In the space of parameters of coefficients of P(x,y) of Theorem 1 we have equivalence classes defined by type-n for n = 0,1,2,3. It is of course possible to get type-4 when $\gamma < -2$ as can easily be seen but we have continued to use the restriction $|\gamma| < 2$ in Lemmas A-M. The boundaries of these equivalence classes define a set $B^{\ell} \subset \mathbb{R}^8$.

Definition B^{ℓ} is the local topological bifurcation set for the Mather unfolding of the germ at the origin of the potential function defined by the polynomial $x^4 + y^4$.

With the assumptions of (i) $P^{-1}(0)$ determining the topological type and (ii) the fractional power series involved have non-zero radius of convergence we can then get the following theorem :

Theorem 2 With assumptions (i) and (ii) stated above, Lemmas A-M give the local topological bifurcation set for $x^4 + y^4$ for all $|\gamma| < 2$. □

The local topological bifurcation set is distinct from the global differential bifurcation set usually studied. This can be seen simply by looking at the (u,v)-diagrams of Chapter III and comparing them with the single point of B^{ℓ} at u = v = 0. A guess might be made that the further splitting of the type-n equivalence classes by the values of the degree of sufficiency would give the local differentiable bifurcation set or something near it. Further it would appear reasonable that the global pictures, topological and differential, differ only because of local considerations. Thus to deduce the topological global bifurcation set from the corresponding differential case could possibly be a matter of this extra splitting.

It is disappointing to note that γ is relevant to Lemmas A-M and hence to B^{ℓ}. The promise that we could ignore γ in the Mather unfolding to get the topological unfolding offered by remarks at the end of §1 is not fulfilled in the local case.

§3 General Properties of the Potential Functions

The critical points the potential function

$$V(x,y) \equiv x^4 + y^4 + \gamma x^2y^2 + \alpha x^2y + \beta xy^2 + wx^2 + \delta xy + ty^2 - ux - vy \quad \text{----(3)}$$

are given by the intersections of the algebraic curves

$$\frac{\partial V}{\partial x} = 4\,x^3 + 2\gamma xy^2 + 2\alpha xy + \beta y^2 + 2wx + \delta y - u = 0\ ; \qquad \text{------} \quad (4)$$

$$\frac{\partial V}{\partial y} = 4y^3 + 2\gamma x^2 y + 2\beta xy + \alpha x^2 + 2ty + \delta x - v = 0\ . \qquad \text{------} \quad (5)$$

<u>Lemma 5</u> If the polynomial (3) has no repeated factor then there are at most 9 distinct critical points.

<u>Proof</u> The curve (4), (5) are both of order 3. Thus using result given by Walker [5, Th 3·1 p.59] if (4) and (5) do not have a common component there are less than $3 \times 3 = 9$ common points. If (4) and (5) have a common component then the polynomials have a common factor and hence [2, p.229] we have (3) with a repeated factor. □

It is possible for there to be an infinite number of critical for V, since

$$V = x^4 - 2x^2y^2 + y^4 = (x^2 - y^2)^2$$

is a possible choice for (3). This gives $x = \pm y$ consisting of critical points. The other troublesome case of §2

$$V = x^2 + 2x^2y^2 + y^2 = (x^2 + y^2)^2$$

has a single critical point at $x = y = 0$.

<u>Lemma 6</u> A common component of (4) and (5) is either a line or a conic.

<u>Proof</u> (4) and (5) cannot be identical for any choice of parameters. Any common component is given by a common factor and this can only be of degree 2 or degree 1. Thus we get result. □

<u>Lemma 7</u> If (4) and (5) have a common real linear factor then $\gamma \leq -2$.

<u>Proof</u> (4) and (5) have a common real linear factor

$\Rightarrow$ $V(x,y)$ has this factor repeated

$\Rightarrow$ the real variety $V(x,y) = 0$ contains an unbounded line.

If $\gamma > -2$ then $|V(x,y)| \to \infty$ as $|(x,y)| \to \infty$ in the real plane, hence as a real variety $V(x,y) = 0$ is bounded. □

<u>Lemma 8</u> If either polynomial (4) or (5) has a linear factor that is not real, then it has a real quadratic factor.

<u>Proof</u> Complex linear factor of real expression gives complex conjugate also a factor. The product of these conjugate factors is a real quadratic.

<u>Lemma 9</u> If (4) and (5) have a common quadratic factor then $V(x,y) \equiv (x^2 + y^2)^2$ or $V(x,y) \equiv (x^2 - y^2)^2$.

<u>Proof</u> (4) and (5) have a common quadratic factor implies that $V(x,y) = (ax^2 + bxy + cy^2 + dx + ey + f)^2$.

Comparing coefficients with (3) we find the only possible cases are those stated. $\square$

<u>Proposition 10</u> For $\gamma > -2$ the potential function $V(x,y)$ given by (3) has at most 9 critical points.

<u>Proof</u> By Lemma 5 the result follows from proving that (4) and (5) do not have a common component, i.e. polynomials do not have common factor. By Lemma 6 any common factor is linear or quadratic. The restriction $\gamma > -2$ and Lemma 7 shows that there cannot be a real linear common factor.

Lemma 8 shows that complex linear factor in common implies real quadratic common factor.

Lemma 9 shows that common quadratic factors occur in two cases one of which does not satisfy $\gamma > -2$.

Thus we have proved that $V(x,y)$ has at most 9 critical points except perhaps when $V(x,y) \equiv (x^2 + y^2)^2$. In this case there is just 1 real point satisfying $V(x\ y) = 0$ and only one critical point. $\square$

<u>Lemma 11</u> For $\gamma > -2$ the potential function $V(x,y)$ given by (1) has at most one maximum point.

<u>Proof</u> Zeeman [7] has suggested the following argument for the $\gamma = 0$ case which we generalise slightly.

Let there by two distinct maxima and join them with a straight line $y = Ax + B$ or $x =$ const. C. Making this substitution in $V(x,y)$ of (1) we get values for $V(x,y)$ along this line

(i) $(1 + \gamma A^2 + A^4)x^4 + (2AB\gamma + 4A^3B + \alpha A + \beta A^2)x^3 +$
$(\gamma B^2 + 6A^2B^2 + 2\beta AB + \alpha B + w + \delta A + tA^2)x^2$
$+ (4AB^3 + \beta B^2 + \delta B + 2tAB - u - Av)x$
$+ (B^4 + tB^2 - vB)$;

(ii) $y^4 + (\gamma C^2 + \beta C + t)y^2 + (\alpha C^2 + \delta C - v)y + (C^4 + 2wC^2 - uC)$.

If $\gamma > -2$ then $(1 + \gamma A^2 + A^4) > 0$ for all real A so that (i) and (ii) are both quartic curves with at most one maximum. Since a maximum for $V(x,y)$ will give a maximum on a linear section we get the required result. $\square$

<u>Lemma 12</u> For $\gamma > -2$ the potential function $V(x,y)$ given by (3) can only have one of the following sets of critical points :

(a) n minima and (n-1) saddles $n = 1,2,3,4,5$;

(b) 1 maximum, n minima and n saddles $n = 1,2,3,4$.

<u>Proof</u> For $\gamma > -2$ for sufficiently large k, $V^{-1}(k)$ will be a simple closed curve. V will thus be topological equivalent to a Morse function on a two-disc with the gradient transversal to the boundary. The boundary can be identified with a point to give a Morse function on S^2 with maximum at this point. Since the Euler characteristic of S^2 is 2, we get from the Morse equality

$$(\text{no. of max.}) - (\text{no. of saddles}) + (\text{no. of min.}) = 2.$$

If the original function $V(x,y)$ has no maximum and there are less than 10 critical points, we get case (a). If V has one maximum we get case (b). Lemma 11 shows that these are the only numbers of maxima for V. $\square$

Although the case of 5 minima and 4 saddle points is topologically possible, it seems to be true that for $V(x,y)$ of (1) this situation cannot occur. To prove this requires detailed use of the algebraic form of the function since it is clearly not ruled out by topological considerations. The detailed reason appears to be that if $V(x,y)$ has 9 critical points, then for at least three of them $\dfrac{\partial^2 V}{\partial x^2} < 0$ which shows that at least 3 cannot be minima. This result has not been established even in the apparently simpler $\gamma = 0$ case where

$$4\beta\frac{\partial V}{\partial x} \equiv (2\alpha x + 2\beta y + \delta)^2 + \left\{16\beta x^3 - 4\alpha^2 x^2 - (4\alpha\delta - 8\beta w)x - (\delta^2 + 4\beta u)\right\} ;$$

$$4\alpha\frac{\partial V}{\partial y} \equiv (2\alpha x + 2\beta y + \delta)^2 + \left\{16\alpha y^3 - 4\beta^2 y^2 - (4\beta\delta - 8\alpha t)y - (\delta^2 + 4\alpha v)\right\}$$

The only cases where the $\dfrac{\partial^2 V}{\partial x^2}$ result clearly holds are those for which $\alpha = \beta = \delta = \gamma = 0$ and the function (3) reduces to

$$V(x,y) = x^4 + y^4 + wx^2 + ty^2 - ux - vy$$

with $$\frac{\partial V}{\partial x} = 4x^3 + 2wx - u ;$$

$$\frac{\partial V}{\partial y} = 4y^3 + 2ty - v.$$

To get nine critical points we have three lines $x = \text{const.}$ for $\dfrac{\partial V}{\partial x} = 0$ and three lines $y = \text{const.}$ for $\dfrac{\partial V}{\partial y} = 0$. Here $\dfrac{\partial^2 V}{\partial x^2} < 0$ on the middle $x = \text{const.}$ line on which occur three of the critical points.

All the other cases of Lemma 12 will occur as can be seen from results of Chapter III.

§4 The Double R-H Catastrophe

It is possible to consider the unfolding of $x^4 + y^4$ in the form

$$V(x,y) \equiv x^4 + (\gamma y^2 + \alpha y + w)x^2 + (\beta y^2 + \delta y - u)x + (y^4 + 2ty^2 - vy) \quad \text{---(6)}$$

and the maxima and minima of this function as a function of x depend only on

$$m = \gamma y^2 + \alpha y + w \text{ ;} \quad \text{-------------(7)}$$

$$n = \beta y^2 + \delta y - u \text{ .} \quad \text{-------------(8)}$$

We can thus consider a (m,n)-plane and look at the type and number of the stationary points of (6) as a function of one variable. In the (m,n)-plane we get the R-H catastrophe curve $4m^3 + 27n^2 = 0$ dividing the 2 minima region from the 1 minimum region. Further we can construct in (m,n,x)-space the surface $4x^3 + 2mx + n = 0$ with regions corresponding to maxima and minima (cf. Zeeman[8]). Also on the (m,n)-plane we have a locus defined by parametric equations (7),(8) and this will be related to $4m^3 + 27n^2 = 0$ in some way. Thus given y we find a point on (7),(8) curve which will give x values from the surface $4x^3 + 2mx + n = 0$. Plotting x against y we get the cubic $\frac{\partial V}{\partial x} = 0$. The maximum region will correspond to $\frac{\partial^2 V}{\partial x^2} < 0$ the minima to $\frac{\partial^2 V}{\partial x^2} > 0$.

If we start from

$$V(x\ y) \equiv y^4 + (\gamma x^2 + \beta x + t)y^2 + (\alpha x^2 + \delta x - v)y + (x^4 + 2wx^2 - wx) \quad \text{--(9)}$$

we get the curve

$$m_1 = \gamma x^2 + \beta x + t \quad \text{-----------(10)}$$

$$n_1 = \alpha x^2 + \delta x - v \quad \text{-----------(11)}$$

in an (m_1, n_1)-plane. From this we can get the $\frac{\partial V}{\partial y} = 0$ curve in terms of the $4y^3 + m_1 y + n_1 = 0$ surface.

Rough results on the intersection of $\frac{\partial V}{\partial y} = 0$ and $\frac{\partial V}{\partial x} = 0$ deduced from drawing these diagrams help to reinforce the conjecture that we can only have 9 critical points if at least 3 correspond to x on the 'maximum' part of $4x^3 + mx + n = 0$ and so cannot be minima of V(x,y) as a function of two variables.

For the cases with $\alpha = \beta = \gamma = \delta = 0$ we find that $\frac{\partial V}{\partial x} = 0$ depends only on x and equations (7),(8) reduce to $m = w$, $n = -u$. The value of the state variable x is thus controlled by a R-H catastrophe with control variables (w,u). Similarly for $\alpha=\beta=\gamma=\delta=0$ we get (10),(11) reducing to $m_1 = t$, $n_1 = -v$ and the state variable y being controlled by (t,v).

Definition 1 (i) The parameters $\alpha, \beta, \gamma, \delta$ in the unfolding of $x^4 + y^4$ we call the interaction parameters;

(ii) The parameters w,u we call the x-control parameters;

(iii) The parameters t,v we call the y-control parameters.

This definition suggests the investigation by computer simulation (and/or otherwise) of the situation in which interaction occurs between A and B. A is allowed to choose w_0, u_0 to get the desired state of x and B is allowed choice of t_0, v_0 to control value of y. The resulting x,y would be calculated from the minima of V(x y) where $\alpha,\beta,\gamma,\delta$ would initially be assumed fixed as parameters of the conditions of interaction. More generally one would expect dynamic models with feedback from x y values to choice of u,v,w,t and perhaps variations in the values of $\alpha,\beta,\gamma,\delta$.

§5 Conclusions

As stated at the beginning of the chapter, this is by way of being a preliminary investigation into some detailed aspects of the catastrophe we have defined as the double Rieman-Hugoniot catastrophe. The individual sections of the chapter give many indications of what could still be done. Further it was suggested in Chapter III that a good deal of the standard unfolding pictures for this catastrophe could be obtained by continuing the line of investigation of Appendix I and Chapter III. In context of this chapter, we see that this method will lead to the differentiable pictures of the global differential unfolding. In §2 of this chapter we were aiming at the differentiable pictures of the local topological unfolding. For §3 of this chapter we could state the aim as topological pictures of the global differential unfolding although we have not proceeded far in this direction. Thom's original requirement [9] was, however, for the differentiable pictures of the global topological unfolding. We see then that we need not only to complete the lines of study started here, but relate them to another to get at the true pictures of the double Riemann-Hugoniot catastrophe.

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THREE DIMENSIONAL PICTURES OF THOM'S PARABOLIC UMBILIC

by A.N. GODWIN

§1 Introduction

Thom [3] has introduced the seven elementary catastrophes in $\mathbb{R}^4$. The most complicated and perhaps the most interesting of these is the last in Thom's list, which he calls the parabolic umbilic. In his book [3] and papers [1,2] he has drawn some qualatative sketches of some two dimensional sections of the surface used in applications of the theory. Here we give more accurate pictures of this surface based on quantative computer analysis.

The parabolic umbilic is given by the formula

$$V = x^2y + y^4 + ty^2 + wx^2 - ux - vy \qquad (1)$$

where we consider V as a potential function

$$V : \mathbb{R}^4 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

with $c = (u, v, w, t) \in \mathbb{R}^4$ a point in the unfolding (or control [5]) space and (x,y) as points in the state space $\mathbb{R}^2$. For each $c \in \mathbb{R}^4$ we have a potential function

$$V_c : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

and for each c we are interested in stationary points of V_c. Most (open dense) $c \in \mathbb{R}^4$ give rise to V_c with isolated stationary points and such c are called <u>regular points</u>, the remaining points belong to the <u>bifurcation set</u>, B ($\subset \mathbb{R}^4$) corresponding to those potentials which have coincidences amongst their stationary points. The main objective of this paper is to analyse B and to draw the intersections of B with the four three-dimensional coordinate hyperplanes of $\mathbb{R}^4$.

The secondary objective is to draw the <u>Maxwell set</u> which we now define. In each component of $\mathbb{R}^4$-B there are 0, 1 or 2 minima and 0 or 1 maxima (saddle points being ignored). The Maxwell convention assigns a preference to the lower minimum if more than one exist. So in those components with two minima we pick out the set of points representing V_c with two equal minima as the Maxwell set. In applications the catastrophe set is important, since this is the set representing discontinuities in space-time. The catastrophe set comprises two types of point; (i) those points of the set B which separate regions that differ by one in the number of minima, (ii) (in some applications) the Maxwell set; and so can be deduced from our diagrams.

§2 Two dimensional sections of B

Stationary points of (1) are given by

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$$

and the set B is given by two stationary points coincident this being defined by the equation

$$\begin{vmatrix} \dfrac{\partial^2 V}{\partial x^2} & \dfrac{\partial^2 V}{\partial x \partial y} \\ \dfrac{\partial^2 V}{\partial x \partial y} & \dfrac{\partial^2 V}{\partial y^2} \end{vmatrix} = 0$$

i.e. Hessian vanishes. Thus we have 3 equations

$$u = 2x\,(y + w) \tag{2}$$

$$v = x^2 + 4y^3 + 2ty \tag{3}$$

$$x^2 = (y + w)\,(6y^2 + t) \tag{4}$$

and the set B is given explicitly by elimination of x, y. This can be written as

$$u^2 = 4(p + w)^3\,(6p^2 + t) \tag{5}$$

$$\text{where } p = \sqrt[3]{\{(v/8 - u^2/32) + \sqrt{[(u^4/16 - u^2v/2 + v^2)/16 + t^3/54]}\}} + \sqrt[3]{\{(v/8 - u^2/32) - \sqrt{[(u^4/16 - u^2v/2 + v^2)/16 + t^3/54]}\}}, \tag{6}$$

or other two roots of the cubic equation (8), so that B appears to consist of 3 parts. The form (5), (6), is however not very easy to use in calculations and so we have used the form

$$u = 2\,[(y + w)^3\,(6y^2 + t)]^{\frac{1}{2}} \tag{7}$$

$$v = 10\,y^3 + 6wy^2 + 3ty + wt \tag{8}$$

which for given w,t is a pair of parametric equations in parameter y for a curve in the u,v plane.

The equations (7), (8) can be seen to give real pairs (u,v) for those values of y for which the quintic expression $(y + w)^3 (6y^2 + t)$ is positive, so we can write down ranges of y relevant to B for any given w,t. Further we can see that the curve will have reflective symmetry in the v-axis. Also in general case we can consider large y and we have (i) $y \to -\infty$ no real points (ii) $y \to +\infty$ $v \sim u^{6/5}$. For the more detailed analysis of B that follows, we need besides (7) and (8) the equations

$$\frac{dv}{dy} = 30y^2 + 12wy + 3t \qquad (9)$$

$$\frac{du}{dy} = \sqrt{[(y + w)/(6y^2 + t)]}\ (30y^2 + 12wy + 3t) \qquad (10)$$

(taking +ve square root and looking at $u \geq 0$) which apart from exceptional points gives

$$\frac{dv}{du} = \sqrt{[(6y^2 + t)/(y + w)]} \qquad (11)$$

We shall further require the expression

$$\frac{d}{dy}\left(\frac{dv}{du}\right) = (6y^2 + 12wy - t)/2(y + w)^{3/2}(6y \ + t)^{\frac{1}{2}} \qquad (12)$$

the +ve square roots applying to $u \geq 0$. Using (7) - (12) we can decide the ranges of values of y for each (u,v) graph and choose suitable w,t values to capture the four dimensional behaviour of B. Further these formulae give a basis for establishing such features as cusps, discontinuities and intersections which need analytic support to computer evidence.

To get at the details of u,v sections we split the discussion into the three cases $t > 0$, $t = 0$, $t < 0$.

(A) $t > 0$

For $t > 0$ we see from (7) that the range of y for which real points occur is the semi-infinite interval $y \geq -w$, the quintic for u having a triple real root at $y = -w$ and two complex roots. From (9), (10), (11) we see that points of interest are $y = -w$, $y = \pm\sqrt{(-t/6)}$ and $y = -w/5 \pm \sqrt{w/25 - t/10}$. The points $y = \pm\sqrt{[-t/6]}$ correspond to imaginary u and we do not consider them further here. $y = -w$ gives $u = 0$, $v = -3w^3 - 2tw$ and by using (i) $\lim_{y \to -w} (dv/du) = +\infty$; (ii) symmetry; we see that we have a cusp on the v-axis, pointing in the v decreasing direction. This will hold for all w.

$y = -w/5 \pm\sqrt{[w^2/25 - t/10]}$ is real for $w^2 \geqslant 5t/2$ (>0), but for such negative w we have

$$-w/5 \pm \sqrt{[w^2/25 - t/10]} < -2w/5 < -w$$

so that these y values give imaginary u. Thus we need only consider $w \geqslant \sqrt{[5t/2]}$, and since for such w

$$-w < -w/5 - \sqrt{[w^2/25 - t/10]} \leqslant -w/5 + \sqrt{[w^2/25 - t/10]} < 0 \qquad (13)$$

these points do not complicate the picture at $y = -w$.

Now from (11) $dv/du > 0$ except for perhaps exceptional points, these being just the points where dv/du is undefined due to the vanishing of $30y^2 + 12wt + 3t$ i.e. $y = -w/5 \pm\sqrt{[w^2/25 - t/10]}$. At these points du/dy and dv/dy both vanish giving a cusp point in the u, v curve. (12) shows that dv/du is decreasing with y for

$$-w-\sqrt{[w^2 + t/6]} < y < -w + \sqrt{[w^2 + t/6]}$$

and since for $w > 0$,

$$-w-\sqrt{[w^2 + t/6]} < -w < 0 < -w + \sqrt{[w^2 + t/6]},$$

dv/du decreases through both cusps (compare (13)) so that branches are on opposite sides of the common tangent. To establish that picture is always as given in Fig. 7 (inset $w = 0.707$) or Fig. 11 (insets $t > 0$) we need only consider the u-values corresponding to the v-value given by $y = -w/5 - \sqrt{[w^2/25 - t/10]}$ and these are calculated as the cusp point and the point given by $y = -w/5 + 2\sqrt{[w^2/25 - t/10]}$ (by use of (8).) Letting $y = (-w/5 + a)$ we find

$$u = 24\left[a^5 + 2wa^4 + (w^2 + t/6)a^3 - 4w(w^2/25 - t/10)a^2\right.$$
$$\left.- \{16w^2(w^2/25 - t/10)/5\} a + 32w^3(2w^2/25 + t/3)/125\right] \qquad (14)$$

points $y = \pm\sqrt{[-t/6]}$ correspond to imaginary u and we do not consider them further here. $y = -w$ gives $u = 0$, $v = -4w^3 - 2tw$ and by using (i) $\lim_{y \to -w} (dv/du) = +\infty$; (ii) symmetry; we see that we have a cusp on the v-axis, pointing in the v decreasing direction. This will hold for all w.

$y = -w/5 \pm\sqrt{[w^2/25 - t/10]}$ is real for $w^2 \geqslant 5t/2$ (>0), but for such negative w we have

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so that these y values give imaginary u. Thus we need only consider $w \geqslant \sqrt{[5t/2]}$, and since for such w

$$-w < -w/5 - \sqrt{[w^2/25 - t/10]} \leqslant -w/5 + \sqrt{[w^2/25 - t/10]} < 0 \qquad (13)$$

these points do not complicate the picture at $y = -w$.

Now from (11) $dv/du > 0$ except for perhaps exceptional points, these being just the points where dv/du is undefined due to the vanishing of $30y^2 + 12wt + 3t$ i.e. $y = -w/5 \pm\sqrt{[w^2/25 - t/10]}$. At these points du/dy and dv/dy both vanish giving a cusp point in the u, v curve. (12) shows that dv/du is decreasing with y for

$$-w - \sqrt{[w^2 + t/6]} < y < -w + \sqrt{[w^2 + t/6]}$$

and since for $w > 0$,

$$-w - \sqrt{[w^2 + t/6]} < -w < 0 < -w + \sqrt{[w^2 + t/6]},$$

dv/du decreases through both cusps (compare (13)) so that branches are on opposite sides of the common tangent. To establish that picture is always as given in Fig. 7 (inset $w = 0.707$) or Fig. 11 (insets $t > 0$) we need only consider the u-values corresponding to the v-value given by $y = -w/5 - \sqrt{[w^2/25 - t/10]}$ and these are calculated as the cusp point and the point given by $y = -w/5 + 2\sqrt{[w^2/25 - t/10]}$ (by use of (8).) Letting $y = (-w/5 + a)$ we find

$$u = 24\Big[a^5 + 2wa^4 + (w^2 + t/6)a^3 - 4w(w^2/25 - t/10)a^2$$
$$- \{16w^2(w^2/25 - t/10)/5\}\, a + 32w^3(2w^2/25 + t/3)/125\Big] \qquad (14)$$

and by putting $a = -\sqrt{[w^2/25 - t/10]}$ and $a = +2\sqrt{[w^2/25-t/10]}$ we will get the u-values on the v=const line. The difference in u-values, the cusppoint being assumed smaller, is given by

$$\Delta(u^2) = 18.24 \cdot b^4.(w + b) \tag{15}$$

where $b = \sqrt{[w^2/25 - t/10]}$. Since $\Delta(u^2) > 0$ for all $t > 0$, $w > 0$, the point on the same v-level as the cusp is outside the cusp and this is sufficient to confirm the diagrams. The (y,w) information for $t > 0$ is summarised in Fig. 1, this being the key diagram for computer calculations showing range of y to be searched and relevant w-values. The results of computer calculations have then been used to construct Fig. 7 and some $t > 0$ results used in other diagrams.

Insert Fig. 1

(B) $t = 0$

For $w > 0$ we have the (u,v) graphs modified from the $t > 0$, $w > \sqrt{[5t/2]}$ case by the drawing together of the lower cusp and its reflection to form the 'beak to beak' configuration, Fig. 6 ($w = 0.5$ inset). This form is predicted by use of (9) and (10) to show that $y = 0, -2w/5$ are cusps, (12) to show dv/du changes to give a two sided cusp at $-2w/5$ and a one sided cusp at $y = 0$. (15) is then used as in the $t > 0$ case to show that the $y = 2w/5$ cusp is inside the curve going to infinity. (11) then shows using the limit as $y \to 0$ that $v = 0$ is tgt to the cusp at (0,0) which completes the demonstration. The point given by $y = -w$ is as in $t > 0$, a downward cusp.

With $t = w = 0$ there is now only the problem of point $u = v = 0$ which by consideration of (11) can be seen to have $v = 0$ as a tangent line to the curve, so we have a parabola like curve.

In the case $w < 0$ we have no real cusp points for $y = -2w/5$ and $y = 0$ is a real isolated point. The $y = -w$ point is a downward cusp.

In Fig. 2 we give the (y,w) diagram showing ranges and important values of y for any w when t = 0, the computer calculated u,v sections have been used to construct Fig. 6.

Insert Fig. 2

(c) $t < 0$

From (7) we see that in this case there can be two ranges of y that give real u, one range being a finite interval giving rise to a bounded part of the (u,v) curve, the other a semi-infinite interval corresponding to curves studied in (A) and (B). The end points of these y ranges depend for each fixed t on w and can be seen clearly in the (y,w) diagram Fig. 3 as straight lines. At the end points of these y- intervals u = 0 and v will be given by $-4w^3 - 2tw$, $\pm(4t/3)\sqrt{[-t/6]}$. These (v,w) curves for fixed t are given in Fig. 4, a cubic with two horizontal tangent lines.

Using results summarised in Figs. 3, 4 and considering dv/du we know the general form of the curve for each w,t but certain features need more detailed analysis. These features are; types of intersection with axis of symmetry u = 0; possible cusps; possible double points.

The downward cusp of (A) and (B) will occur at $y = -w$, $v = 4w^3 - 2tw$ and the horizontal, v = const tangent, will occur at the other two points $y = \pm\sqrt{[-t/6]}$, $v = \pm(4t/3)\sqrt{[-t/6]}$ except in cases of coincidence amongst these y values. This can be proved by using limits of dv/du given by (11). If $w = -\sqrt{[-t/6]}$ then use of (11) shows that the downward cusp becomes a finite angle, $2\tan^{-1}(\sqrt{[-12w]})$ bisected by the v-axis. This point is called a <u>hyperbolic point</u> since as w increases through $w = -\sqrt{[-t/6]}$ we get in the nhd. of the intersection for the finite part the transition curve → angle → cusp and for the unbounded branch the opposite sequence. (cf. Zeeman [4]). For the other possible coincidence $w = \sqrt{[-t/6]}$ the bounded part of the curve becomes a distinct isolated point $(0, 8w^3)$, this point will be called an

<u>elliptic point</u> since locally the finite part of the curve is Thom's elliptic umbilic .

(9), (10) again give possible cusps at $y = -w/5 \pm \sqrt{[w^2/25 - t/10]}$ and where this corresponds to real u, these values are marked on Fig. 3. Using this diagram or obvious inequalities of type used in (A) and (B) we can establish that bounded part of curve will have the one real cusp at $y = -w/5 - \sqrt{[w^2/25 - t/10]}$ except at the elliptic point. (This, of course, becomes two real cusps by reflection in the v-axis). (12) shows that for $w < -\sqrt{[-t/6]}$ there are two inflexion points given by $y = -w \pm \sqrt{[w^2 + t/6]}$ and no inflexions for other w. These y values have been drawn on Fig. 3 and this shows clearly that one inflexion is on the bounded part of the curve and one on the unbounded section. The computer drawings of each branch can thus be completely interpreted since (11) and results quoted so far show that the finite and unbounded parts cannot separately produce a double point. Taken together, however, a double point can appear as an intersection of the bounded and unbounded parts of the (u,v) curve. For $-\sqrt{[-t/6]} \leq w \leq \sqrt{[-t/6]}$ use of (11) and (12) is sufficient to show that intersection occurs only when cusp of finite part is outside the other branch and using (15) to compare u values we see that this happens for $w < -\sqrt{[-5t/48]}$ the cusp point actually being on the other branch at $w = -\sqrt{[-5t/48]}$. For $w > \sqrt{[-t/6]}$ using (15) and gradient, inflexion reasoning we have cusps inside and get transverse intersection for $w \geq 2\sqrt{[-t/6]}$ see Fig. 5. For $w < -2\sqrt{[-t/6]}$ we get no double points. For $w = -2\sqrt{[-t/6]}$ we have a double point on the axis at $(0,(-4t/3)\sqrt{[-t/6]})$. The remaining range of w is $(-2\sqrt{[-t/6]}, -\sqrt{[-t/6]})$ which from computer drawings should have one double point and to prove that this is so, formulae so far considered are not sufficient. But formulae and gradient inflexion arguments work for $w = -\sqrt{[-t/6]}$ and by implicit differentiation.

$$\frac{\partial v}{\partial w}\bigg|_{u\,=\,\text{const}} = -2(6y^2 + t) \qquad (16)$$

for $y \neq -w/5 \pm \sqrt{[w^2/25 - t/10]}$ so that for any given u the unbounded part moves upwards as w decreases and the bounded part moves downwards. Thus the unbounded part remains between the cusps and only intersects as given by the computer drawn graphs.

Insert Fig. 3

§3 State Space, Component Types and Maxwell Set

Assuming the components of $\mathbb{R}^4 - B$ contain only potentials with same number and type of stationary points we may define the type of each component by its set of stationary points. Given a point in $\mathbb{R}^4 - B$ we may ascertain its type by first solving (7) and (8) for stationary points (x, y) and then use the Hessian at these points to classify them. Since only one point for each component is required to label the regions of the 3 dimensional pictures, calculations were done for various w and v with u = 0 and t = -6. Sufficient points were chosen so that all regions of the three dimensional pictures in Fig. 5-16 could be labelled. These calculations showed that it was possible to have the following stationary sets - 1 saddle point, 1 minimum and 2 saddle points, 2 minima and 3 saddle points, 1 maximum and 2 saddle points or 1 maximum, 1 minimum and 3 saddle points, the quintic equation for stationary point x values confirming the upper limit of 5 points in these patterns. Since the saddle points are not used in the applications of the theory, only the proper stationary points are referred to in the diagrams.

Zeeman has pointed out to me that turning the potential upside down will interchange maxima and minima which might be useful in some applications. This changed situation can, of course, be easily read off the diagrams presented in this paper. In fact, to relate most closely to the hyperbolic and elliptic umbilic where there can only be one attractor state, we should look at the present pictures this way round.

We are thus in a position to find from the figures the points of the catastrophe set which also are part of B.

To find the Maxwell set, which was defined in §1, we need further calculations in the region of $\mathbb{R}^4$-B with 2 minima. A computer program was written to scan this component for a fixed w, since as can be seen from Fig. 11, this case will give a good indication of its behaviour. The outputs from the program were the details of the values, positions and types of the stationary points for each relevant value of t, u, v. To keep computing time low, a fairly large mesh was used and interpolation for Maxwell set done by hand. The results of these fairly rough calculations are indicated on the relevant inset figures with diagrams of §4.

§ 4 Diagrams

The first set of diagrams, in Fig. 4-7, give 3 dimensional pictures of sections of B by hyperplanes t = const. Fig. 4 is a 'control' diagram for Fig. 5-7 showing that these figures can be roughly summarised by the behaviour of the cubic curves $v = -4w^3-2tw$. The three types of configuration for the three dimensional section are (i) any $t<0$, (ii) $t = 0$, (iii) any $t>0$, The particular choices of $t = -0.3$ for (i) and $t = 0.05$ for (iii) were made for reasons of relative scale of feature.

On the three dimensional pictures the labels concentrate on geometric features such as cusp edges, lines of in section of different parts of B and some special points. Of these special points the elliptic and hyperbolic points have already been defined in §2. The parabolic point then seems a natural name for the origin of coordinates. The name swallow-tail point has given to a pair of points because of the similarity of three dimensional neighbourhoods to one of simpler catastrophy diagrams, Thom [3]. Further certain transverse points where a cusp edge cuts another part of B have been indicated. The two dimensional inset diagrams include the component type information and the Maxwell set and are sections of the three-dimension diagram in the same figure. Some of the special points have also been indicated to aid tying together the main diagram with its sections. The inset (u,v) diagrams are copies of computer drawn diagrams which together with the analysis of §2 were used in visualising the three-dimensional surfaces. To get some idea of the four-dimensional B we can consider the 'controlling' cubic varying with t.

The next set of diagrams Figs. 8-11 are arranged on the same basis as the first set of diagrams Figs. 4-7 except that w is now the fourth or external variable and t is included in the pictures. The 'control' diagram is now Fig. 8, the fixed curve $v = \pm(4t/3)\sqrt{(-t/6)}$ with a variable tangent line. In Fig. 10 (w = 0) the results of extra analysis, showing when turned into pictures, that finite part of the (u,v) - section flattens out at $t \to 0-$ is summarised in a general $t < 0$ section. Otherwise insets and main diagrams are related as described above.

The next pair of diagrams Fig. 12, 13 look at B with u the external variable, but here Fig. 13 is obtained by pertubation of Fig. 12 and the insets are visual deductions from earlier diagrams. Symmetry with respect to u means $u > 0$ same as $u < 0$.

The final set of pictures, Fig. 14-16, have except for a few well documented points a large element of visual deduction from previous figures and do not necessarily have the accuracy of the earlier diagrams. For these figures v is the external variable.

Insert Figs. 4-16.

§5 Conclusions

With this paper there are now fairly detailed pictures available for the discontinuity surface for all of Thom's seven elementary catastrophes. However, Thom himself has used cross-sections of the bifurcation surface of the parabolic umbilic (mark 2).

$$V = x^2y + (x^4 + y^4)/4 + ty^2 + wx^2 - ux - vy$$

in his book [3] and his later applications of the theory to the subject of linguistics. The diagrams corresponding to those of this paper for this new family of potentials show many interesting new features. The author hopes to present these in a subsequent paper.

Acknowledgements

The author wishes to thank E.C. Zeeman for many helpful discussions of the work involved in this paper.

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RUGBY.

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[4] E.C. Zeeman — Breaking of Waves - Proceeding of Symposium on Differential Equations and Dynamical Systems, vol. I Maths. Institute, University of Warwick, p.5.

[5] E.C. Zeeman — Catastrophe Machines - Proceedings of Liverpool Symposium on Singularities, 1970 (to appear).

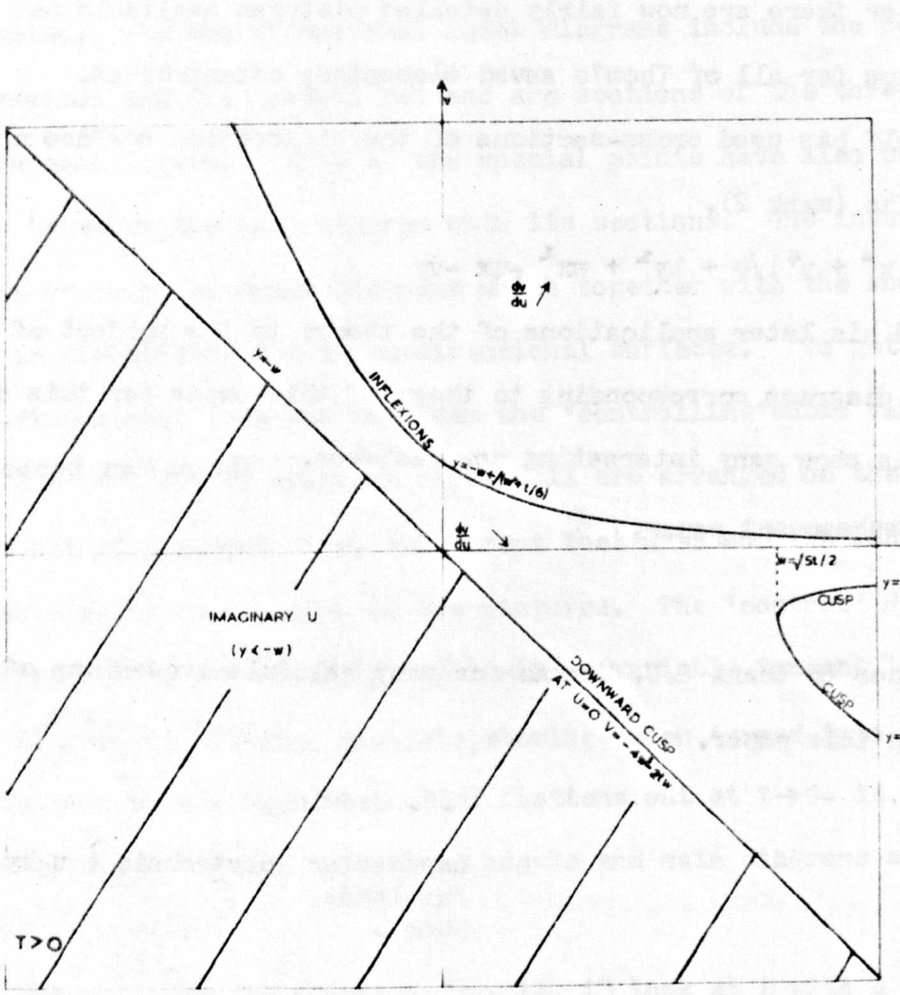

v
dv/du
y=-w
INFLEXIONS
y=-w+√(w²+t/6)
dv/du
w
w=√5t/2
y=-w/5+√(w²/25-t/10)
CUSP
CUSP
y=-w/5-√(w²/25-t/10)
IMAGINARY U
(y<-w)
DOWNWARD CUSP
AT U=O V=-4w³-2tw
T>O

FIG I.

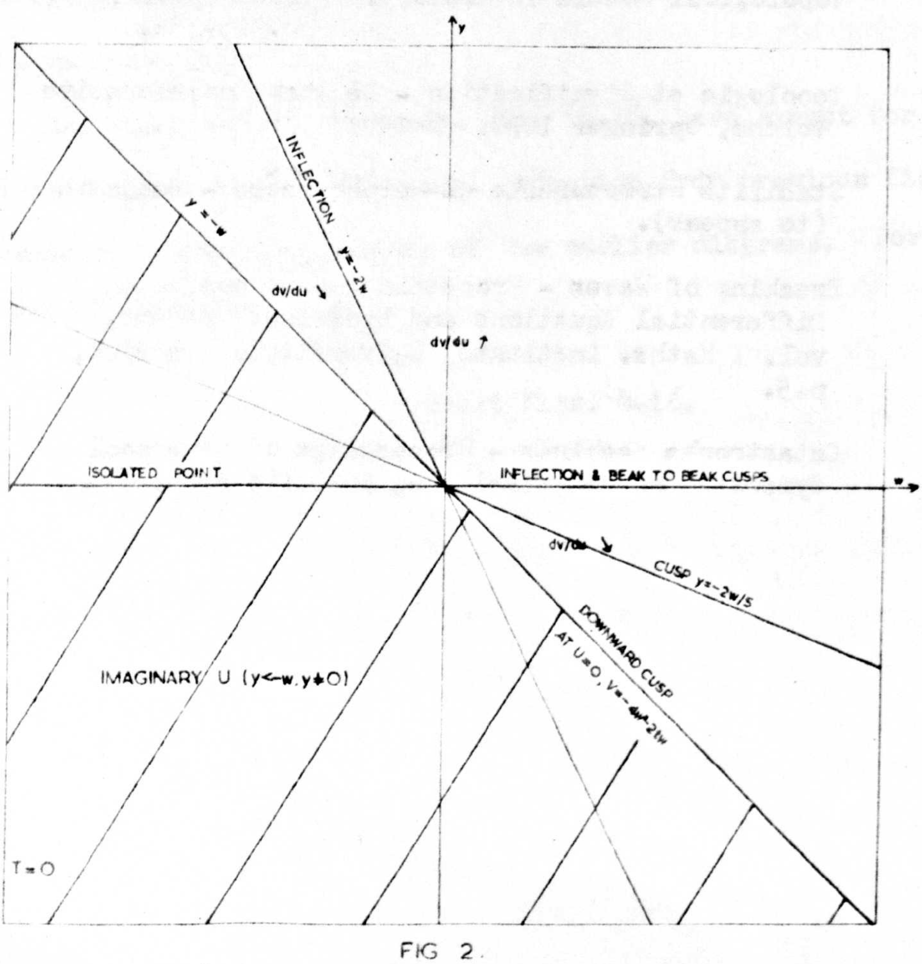

y
INFLECTION
y=-2w
y=-w
dv/du
dv/du
ISOLATED POINT
INFLECTION & BEAK TO BEAK CUSPS
w
dv/du
CUSP y=-2w/5
DOWNWARD CUSP
AT U=O, V=-4w³-2tw
IMAGINARY U (y<-w, y≠O)
T=O

FIG 2.

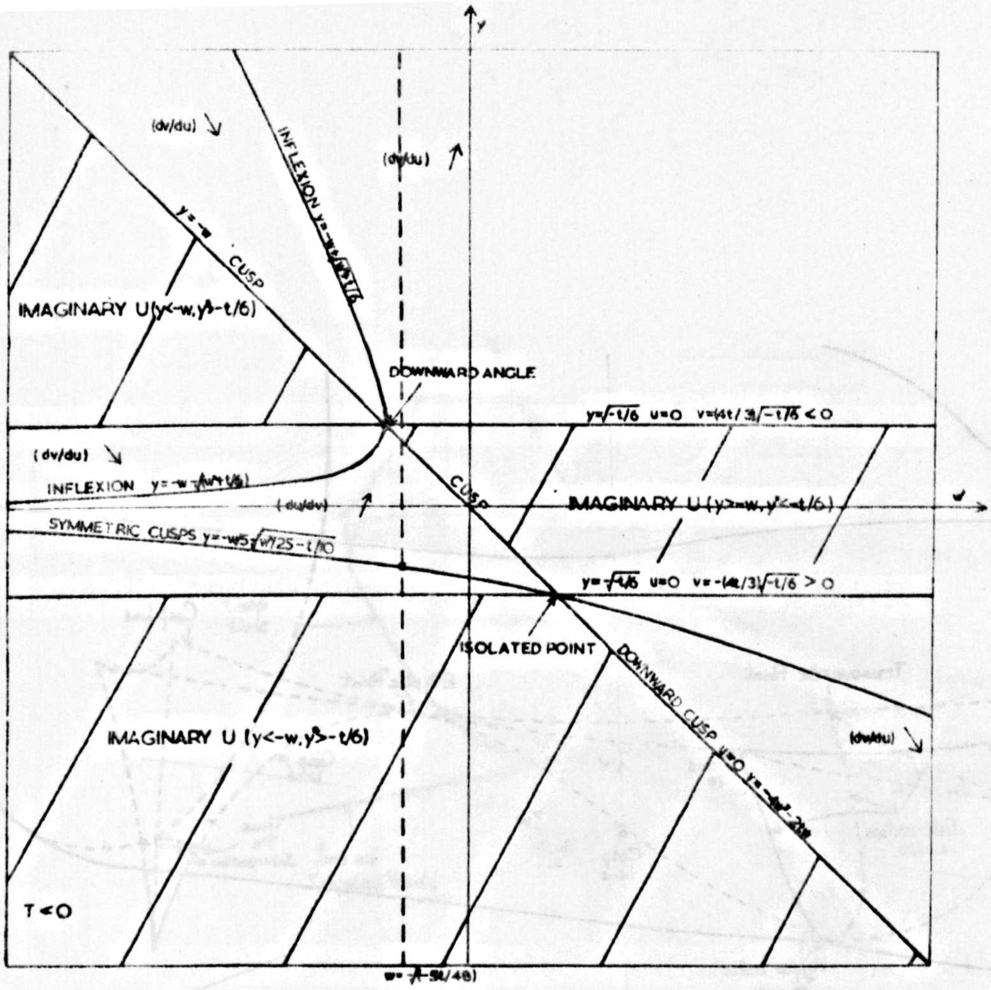

y
(dv/du)
(dv/du)
INFLEXION
CUSP
IMAGINARY U(y<-w, y²>-t/6)
DOWNWARD ANGLE
y=√-t/6 u=0 v=(4t/3)√-t/6 < 0
(dv/du)
INFLEXION
(du/dv)
CUSP
IMAGINARY U (y>-w, y²<-t/6)
w
SYMMETRIC CUSPS
y=-√-t/6 u=0 v=-(4t/3)√-t/6 > 0
ISOLATED POINT
DOWNWARD CUSP
IMAGINARY U (y<-w, y²>-t/6)
(dv/du)
T<0

FIG 3.

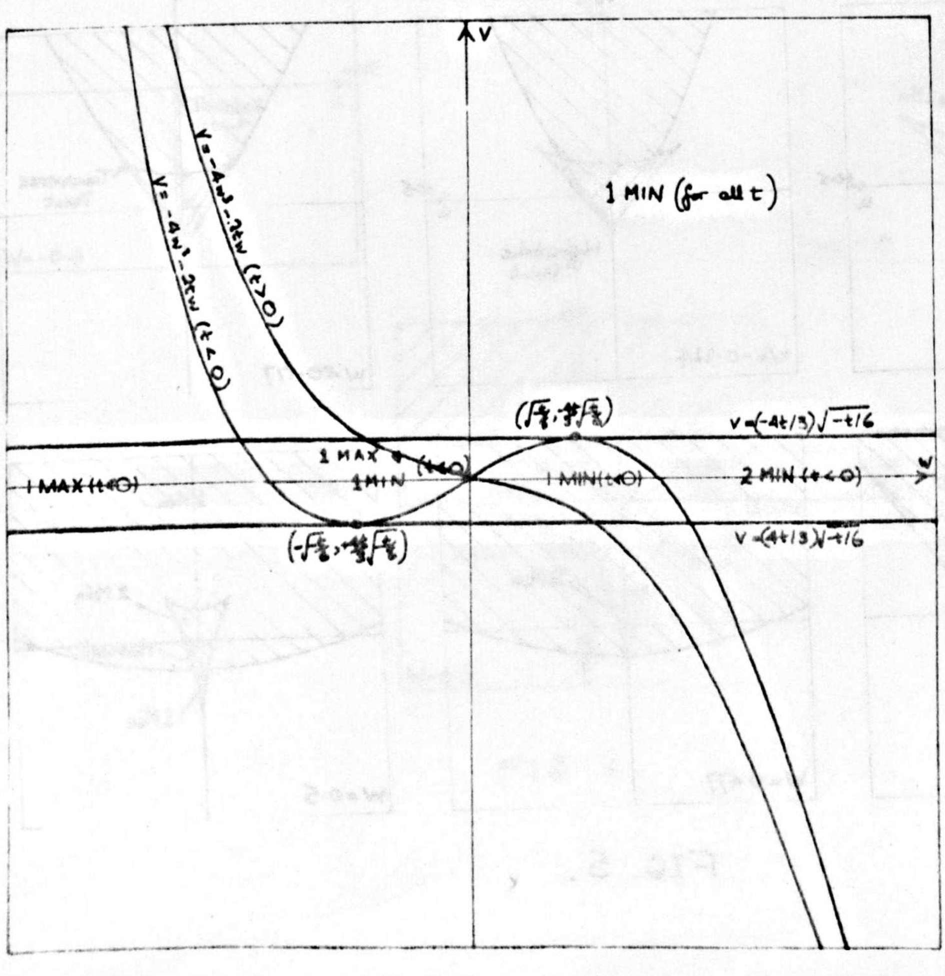
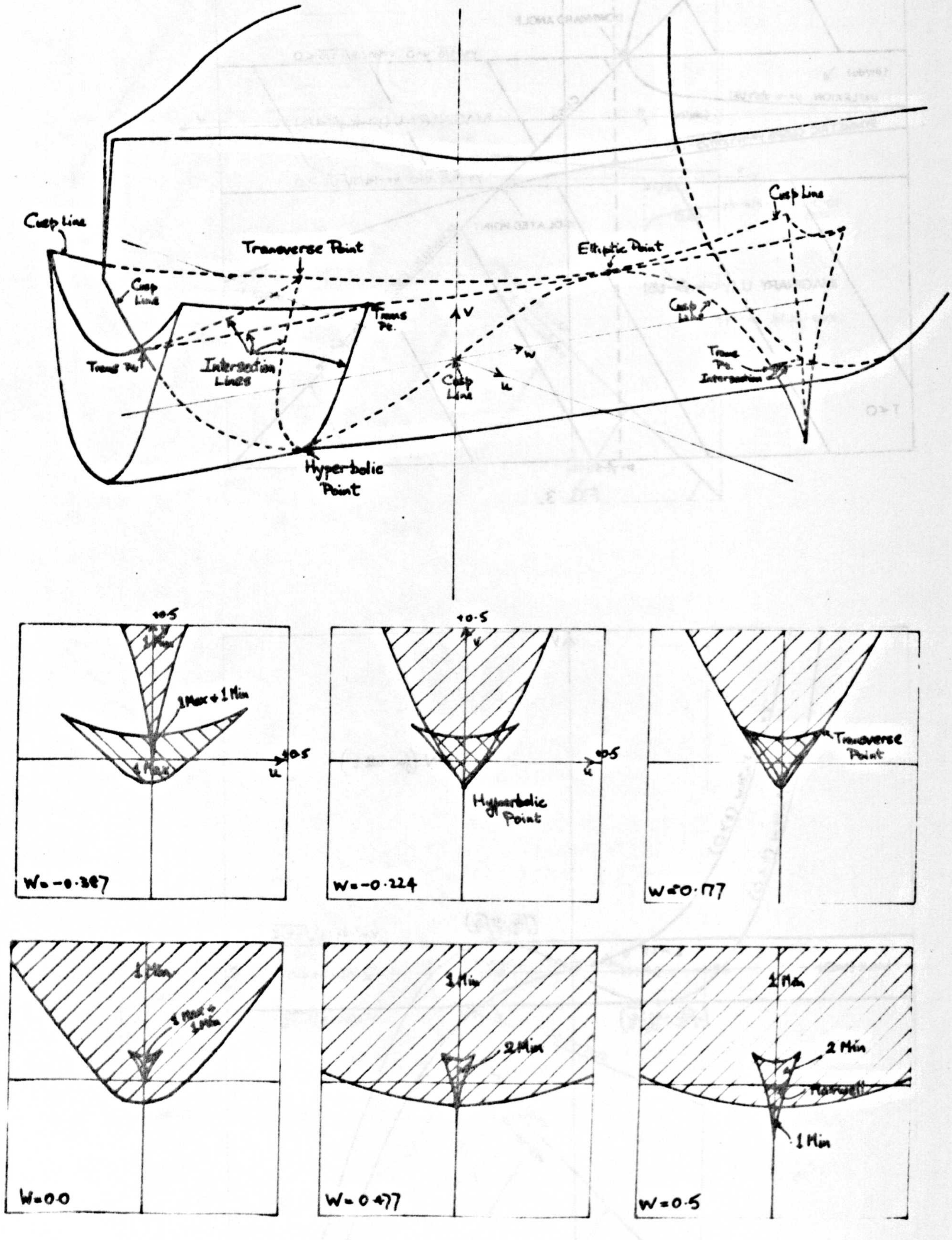

v
1 MIN (for all t)
v=-4w³-2tw (t>0)
v=-4w³-2tw (t<0)
v=(-4t/3)√-t/6
1 MAX & 1MIN (t<0)
1 MAX (t<0)
1 MIN (t<0)
2 MIN (t<0)
w
v=(4t/3)√-t/6

FIG 4

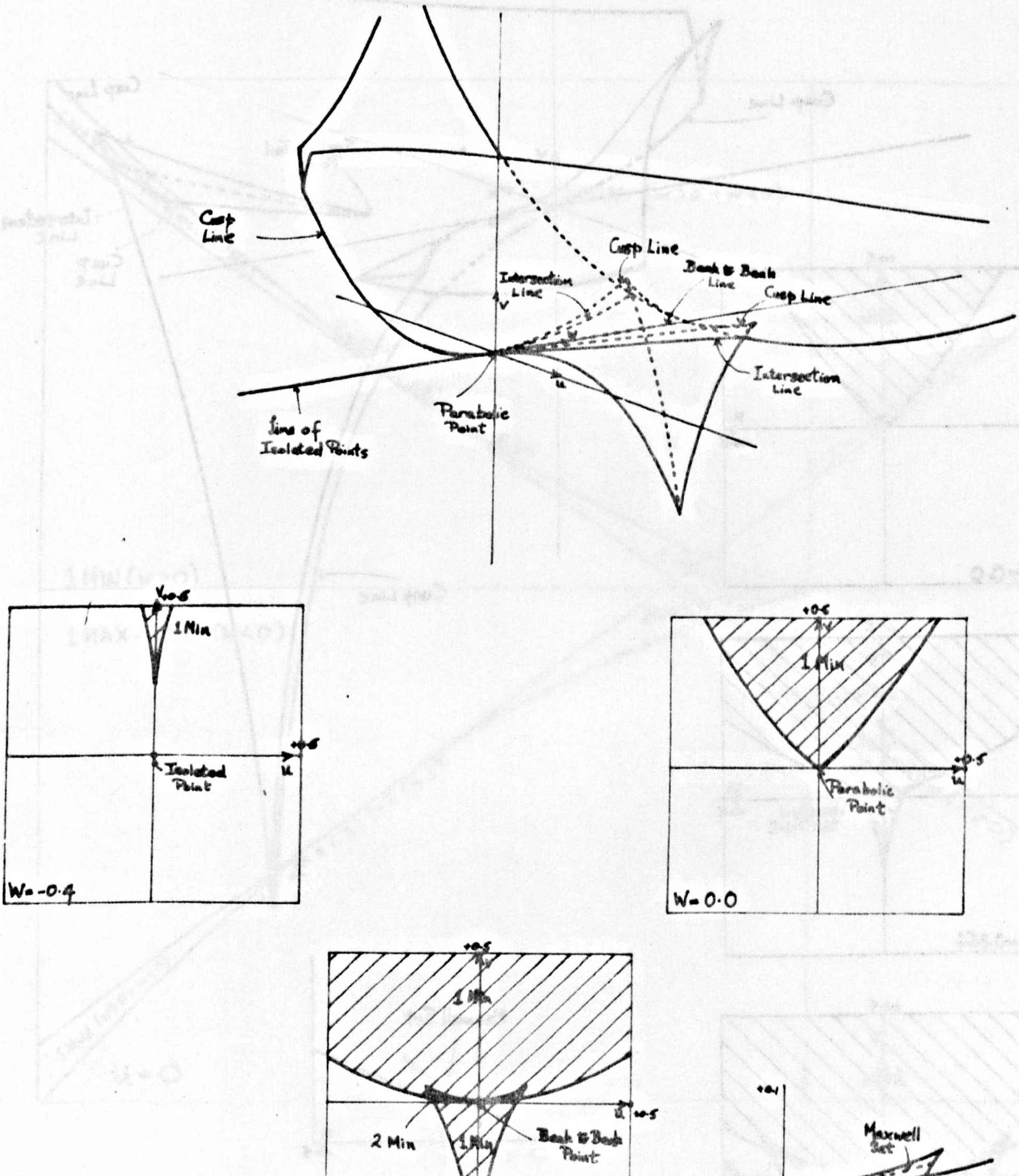
T = -0·3



Cusp Line
Transverse Point
Elliptic Point
Cusp Line
Cusp Line
Trans Pt.
Trans Pt.
Intersection Lines
V
W
u
Cusp Line
Cusp Line
Trans Pt. Intersection
Hyperbolic Point
+0·5
1 Min
1 Max + 1 Min
1 Max
u
W = -0·387
+0·5
V
u
Hyperbolic Point
W = -0·224
Transverse Point
W = 0·177
1 Min
1 Max + 1 Min
W = 0·0
1 Min
2 Min
W = 0·477
1 Min
2 Min
Maxwell
1 Min
W = 0·5

FIG 5.

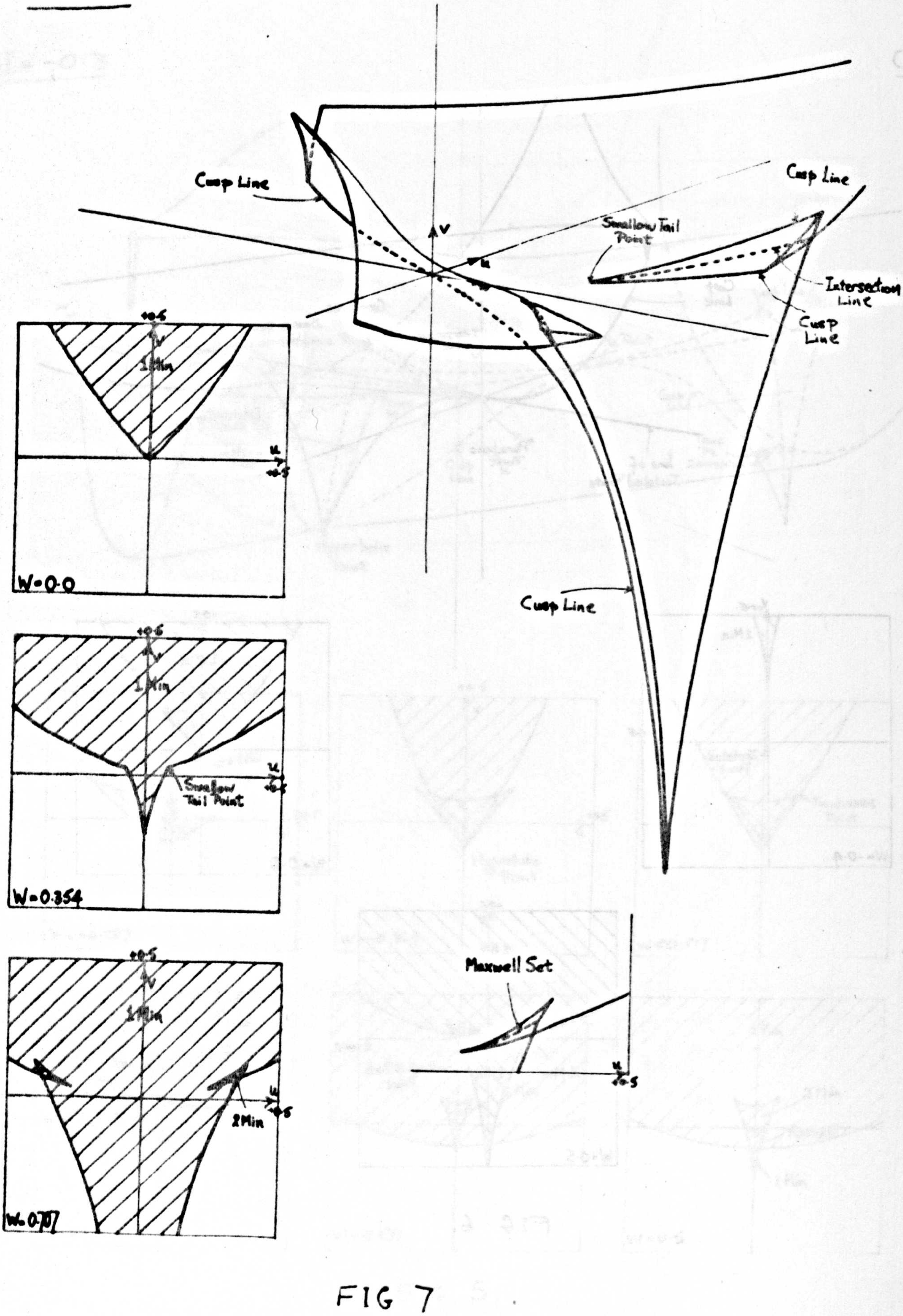
T = 0·0



Cusp Line
Cusp Line
Back & Back Line
Intersection Line
Cusp Line
v
u
Intersection Line
Parabolic Point
Line of Isolated Points
1 Min
Isolated Point
W = -0·4
1 Min
Parabolic Point
W = 0·0
1 Min
2 Min
1 Min
Back & Back Point
W = 0·5
Maxwell Set
+0·1
0·0
+0·2

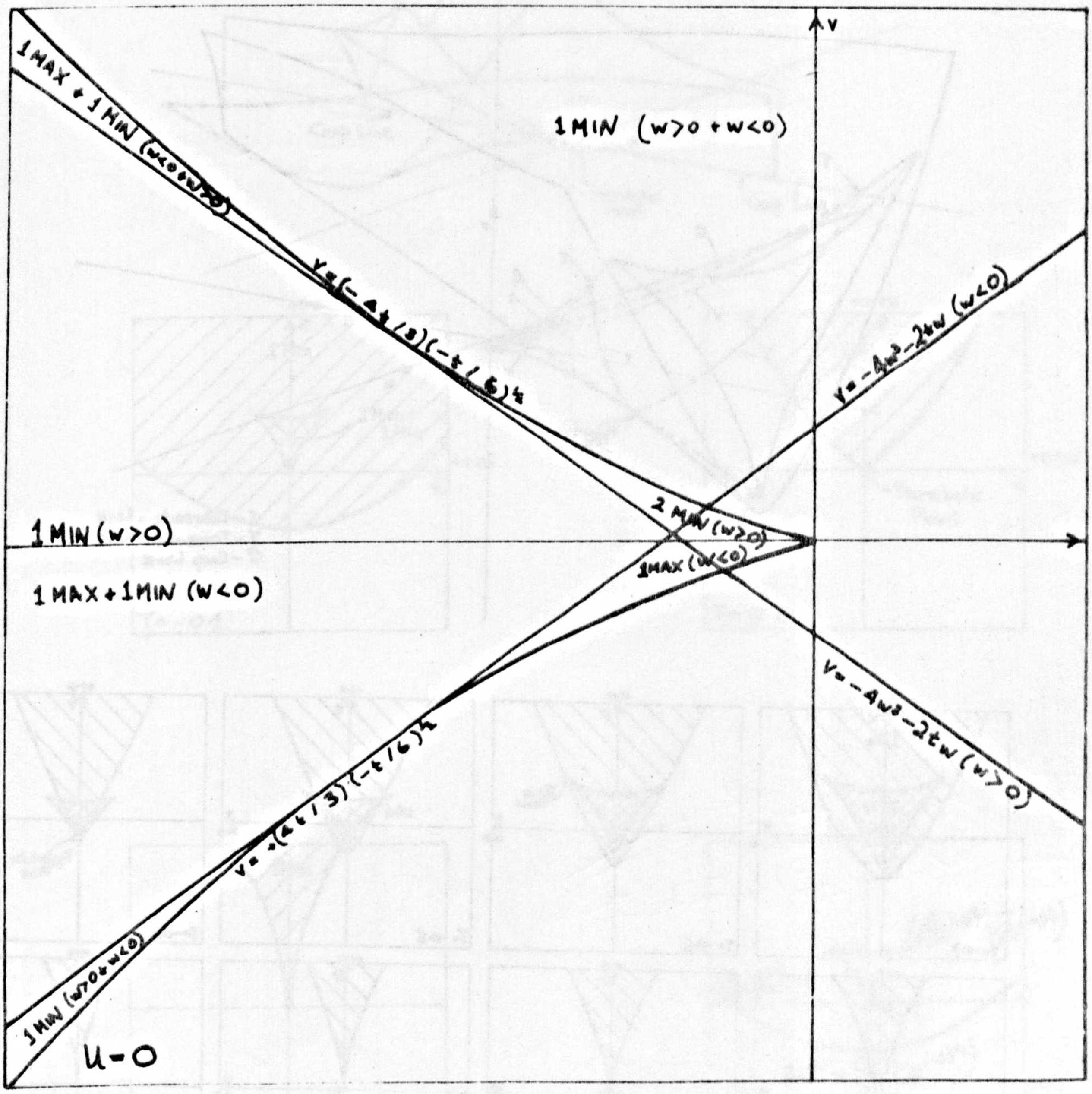
FIG 6.

T = 0·05



Cusp Line
Cusp Line
Swallow Tail Point
Intersection Line
Cusp Line
Cusp Line
1 Min
W=0·0
1 Min
Swallow Tail Point
W=0.354
Maxwell Set
1 Min
2 Min
W=0.707

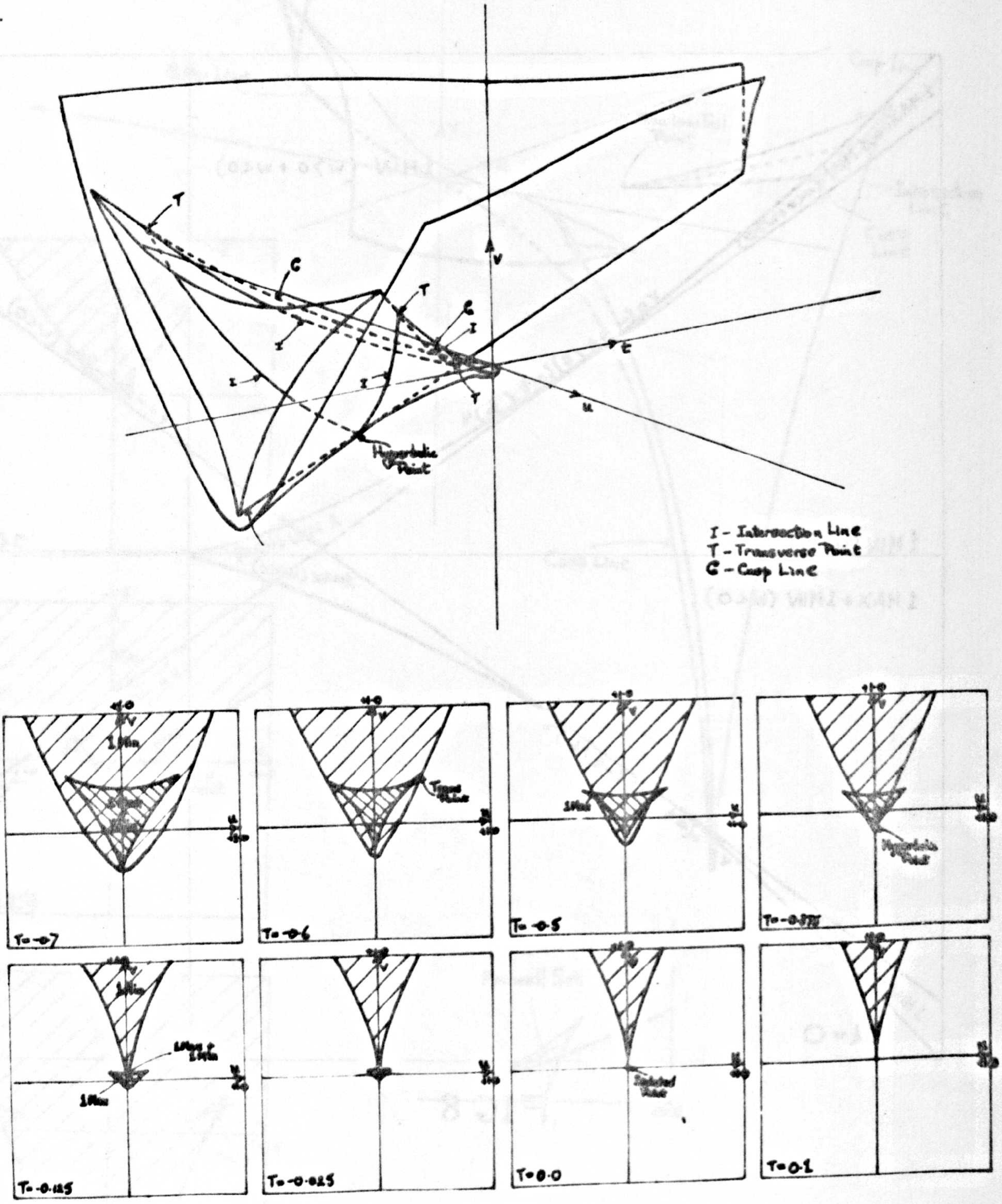
FIG 7



v
1 MAX + 1 MIN (w<0+w>0)
v=(-4t/3)(-t/6)½
1 MIN (w>0 + w<0)
v=-4w³-2tw (w<0)
2 MIN (w>0)
1 MAX (w<0)
1 MIN (w>0)
1 MAX + 1 MIN (w<0)
v=-4w³-2tw (w>0)
v=+(4t/3)(-t/6)½
1 MIN (w>0+w<0)
u-0

FIG 8

W = -0.25



I - Intersection Line
T - Transverse Point
C - Cusp Line
Hyperbolic Point
T = -0.7
T = -0.6
T = -0.5
T = -0.125
T = -0.025
T = 0.0
T = 0.1

FIG 9.

W = 0

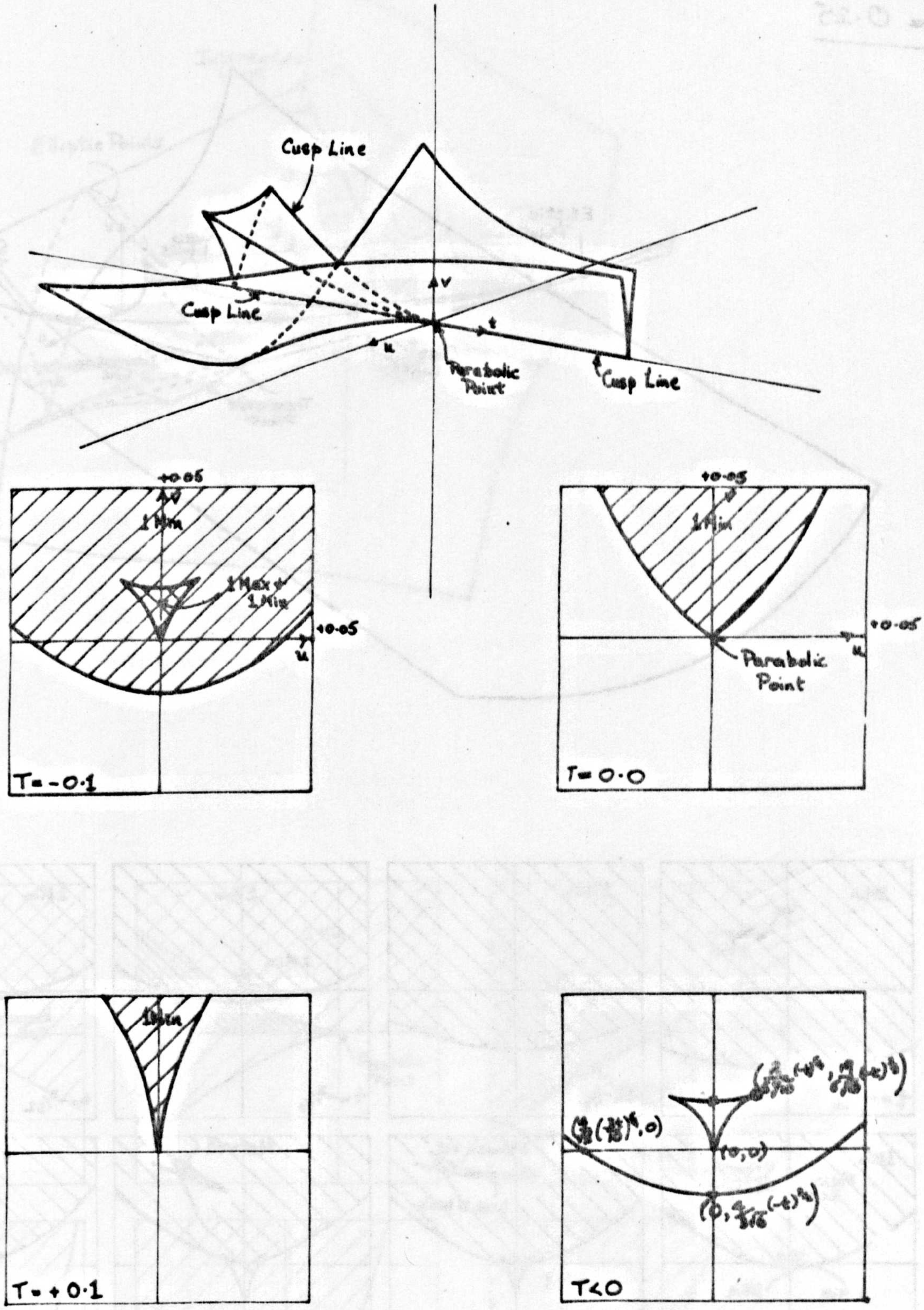
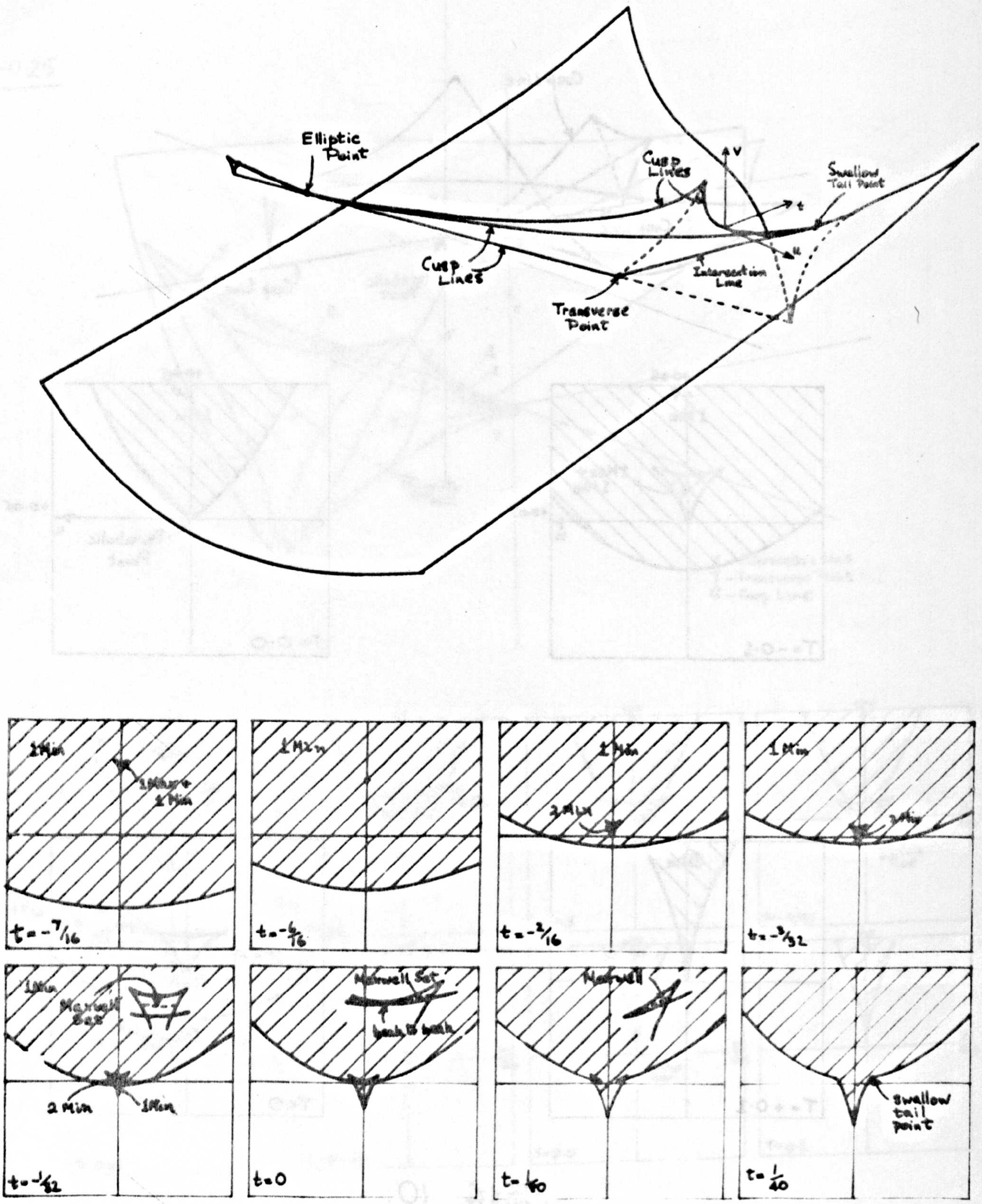

Cusp Line
Cusp Line
Cusp Line
v
t
u
Parabolic Point
+0.05
v
1 Min
1 Max + 1 Min
+0.05
u
T = -0.1
+0.05
1 Min
+0.05
u
Parabolic Point
T = 0.0
1 Min
T = +0.1
(0, 0)
T<0

FIG 10.

W = 0.25



Elliptic Point
Cusp Lines
V
t
u
Swallow Tail Point
Cusp Lines
Intersection Line
Transverse Point
2 Min
1 Max + 1 Min
t = -7/16
1 Min
t = -6/16
1 Min
2 Min
t = -2/16
1 Min
2 Min
t = -3/32
1 Min
Maxwell Set
2 Min
1 Min
t = -1/32
Maxwell Set
back to back
t = 0
Maxwell
t = 1/80
swallow tail point
t = 1/40

FIG 11.

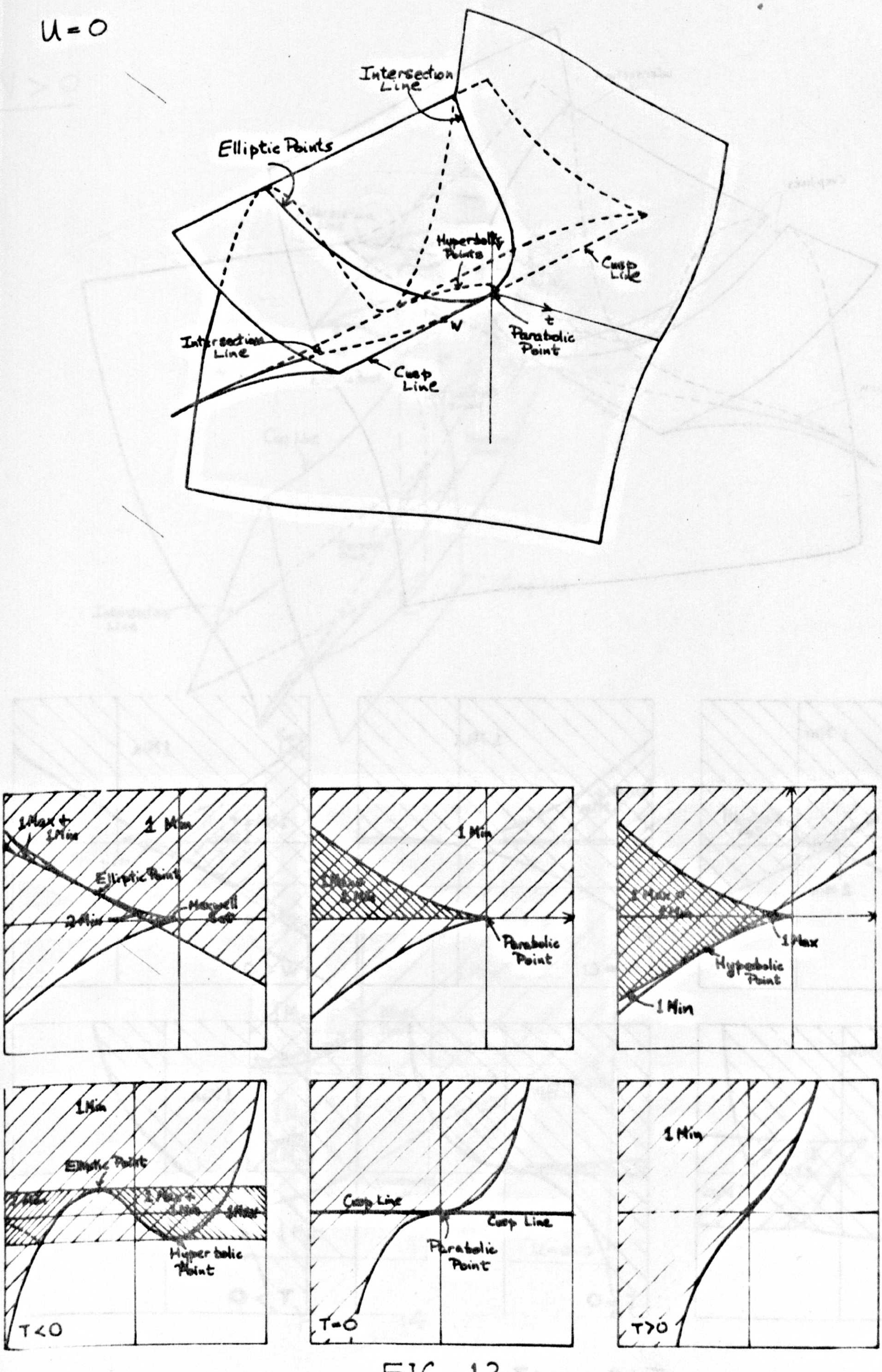
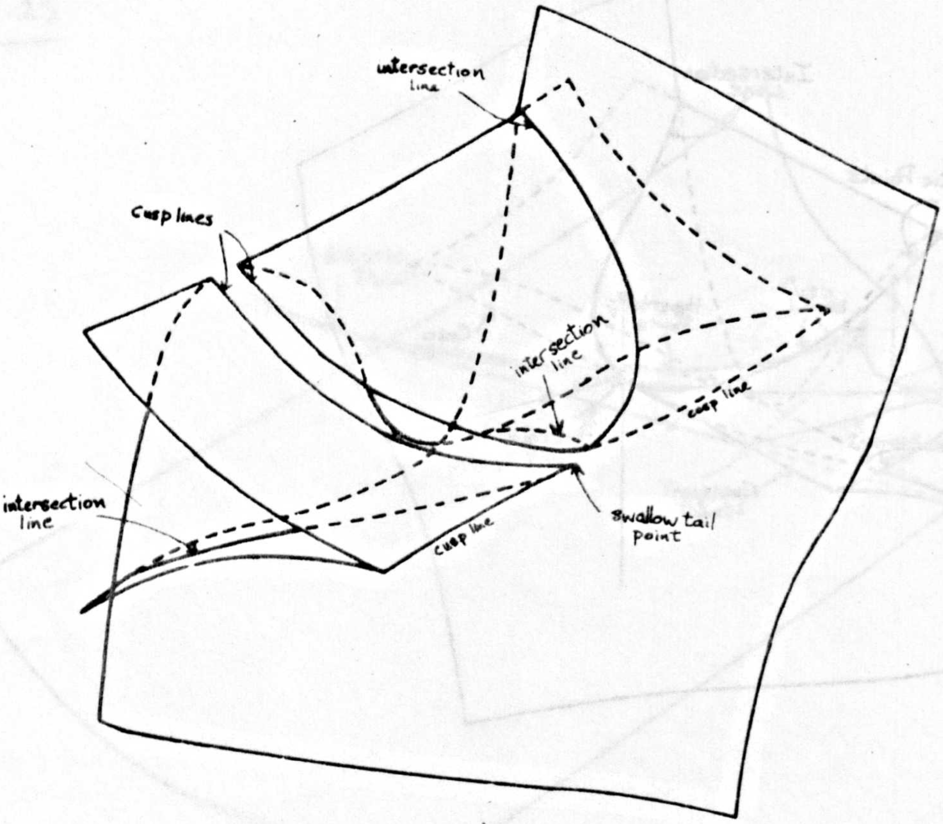
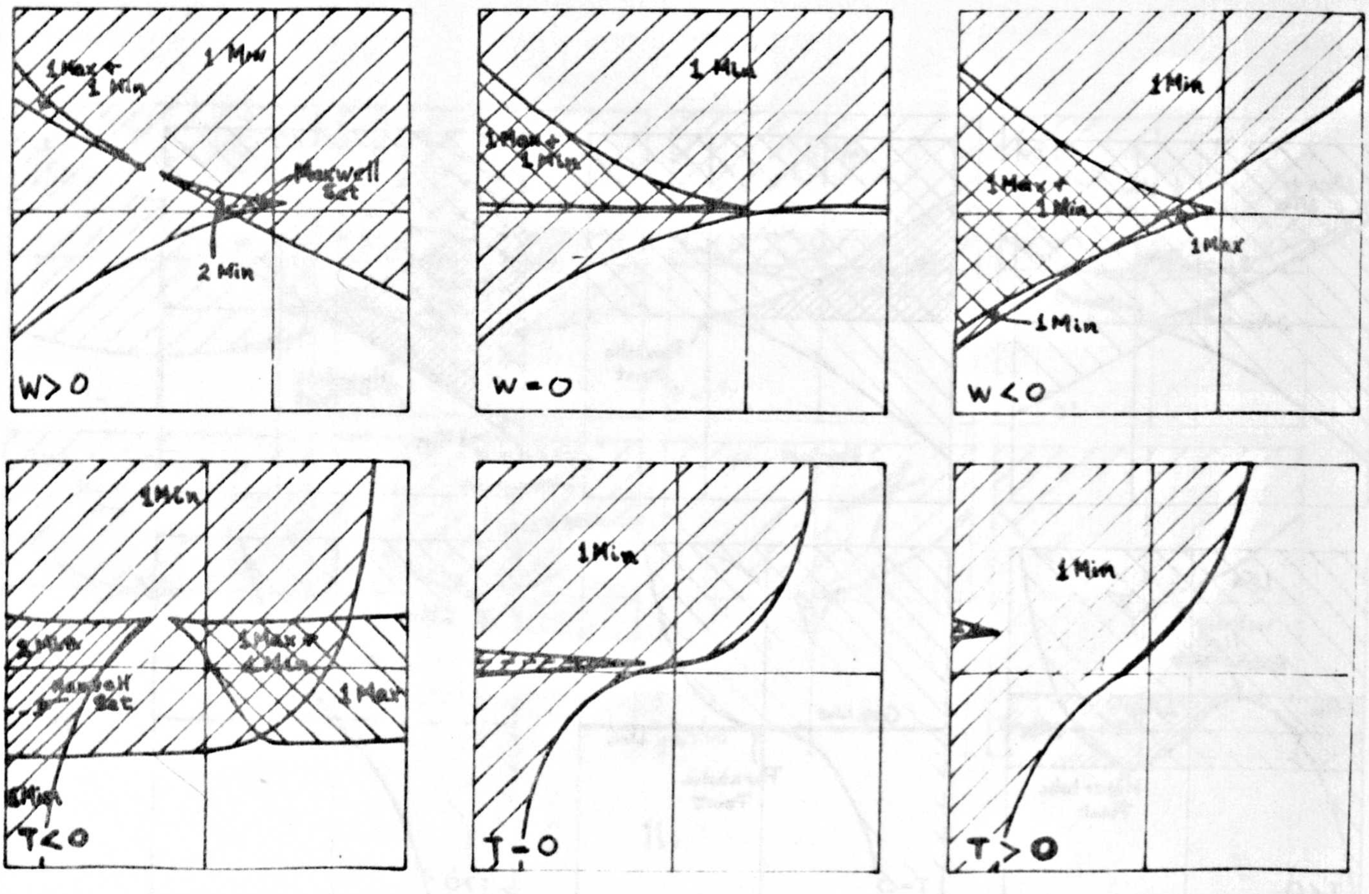

U=0
Intersection Line
Elliptic Points
Hyperbolic Points
Cusp Line
t
w
Parabolic Point
Intersection Line
Cusp Line
1 Max + 1 Min
1 Min
Elliptic Point
2 Min
Maxwell Set
1 Min
Parabolic Point
1 Max
Hyperbolic Point
1 Min
1 Min
Elliptic Point
1 Max + 1 Min
Hyperbolic Point
T<0
Cusp Line
Cusp Line
Parabolic Point
T=0
1 Min
T>0

FIG 12.

U ≠ O



intersection line
cusp lines
intersection line
cusp line
intersection line
cusp line
swallow tail point



1 Max + 1 Min
1 Min
Maxwell Set
2 Min
W > O
1 Min
1 Max + 1 Min
W = O
1 Min
1 Max + 1 Min
1 Max
1 Min
W < O
1 Min
1 Max + 2 Min
Maxwell Set
1 Max
T < O
1 Min
T = O
1 Min
T > O

FIG 13.

V > 0

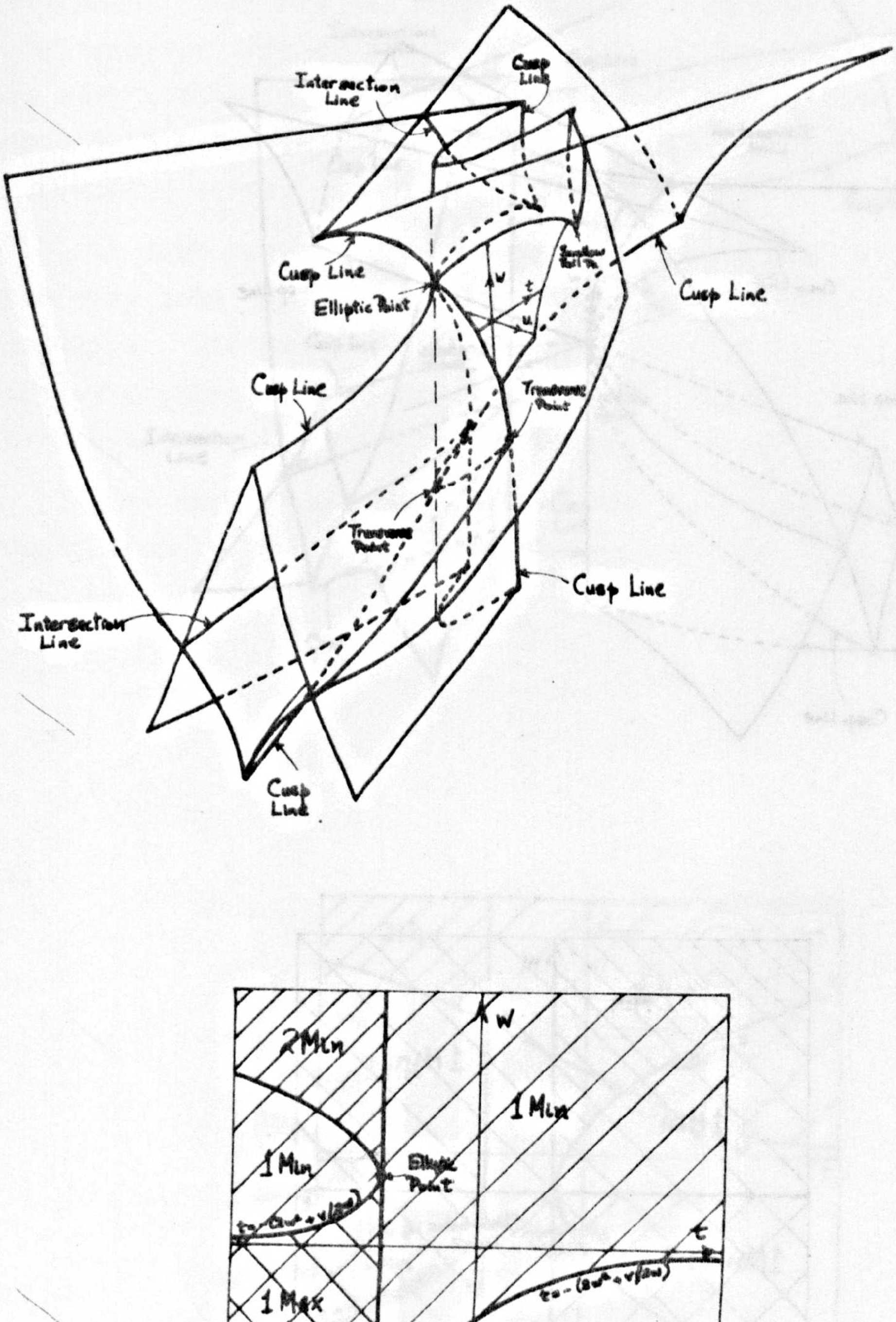
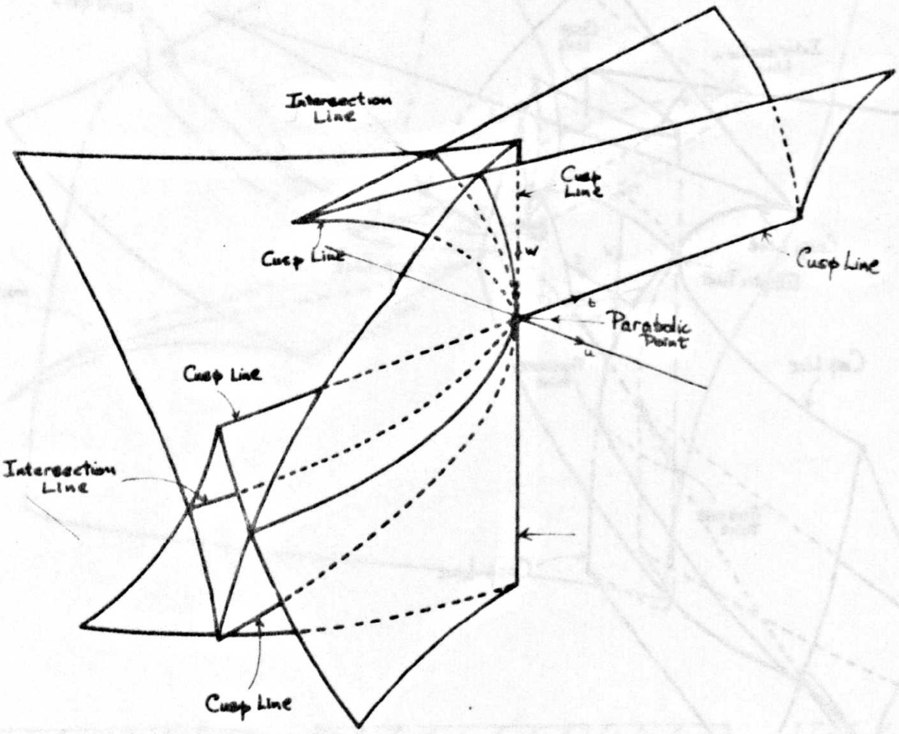

Intersection Line
Cusp Line
Cusp Line
Elliptic Point
Swallow Tail Pt
w
t
u
Cusp Line
Cusp Line
Transverse Point
Transverse Point
Cusp Line
Intersection Line
Cusp Line
w
2 Min
1 Min
1 Min
Elliptic Point
t
1 Max
+
1 Min
Transverse Point
1 Max
u=0.0

FIG 14

V = 0


Intersection Line
Cusp Line
w
Cusp Line
Cusp Line
t
Parabolic Point
u
Cusp Line
Intersection Line
Cusp Line

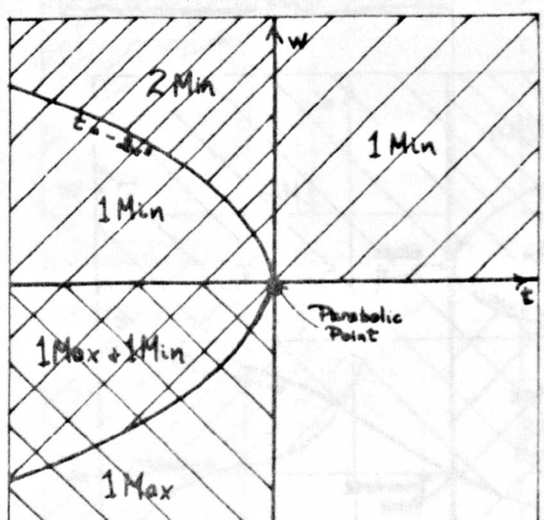
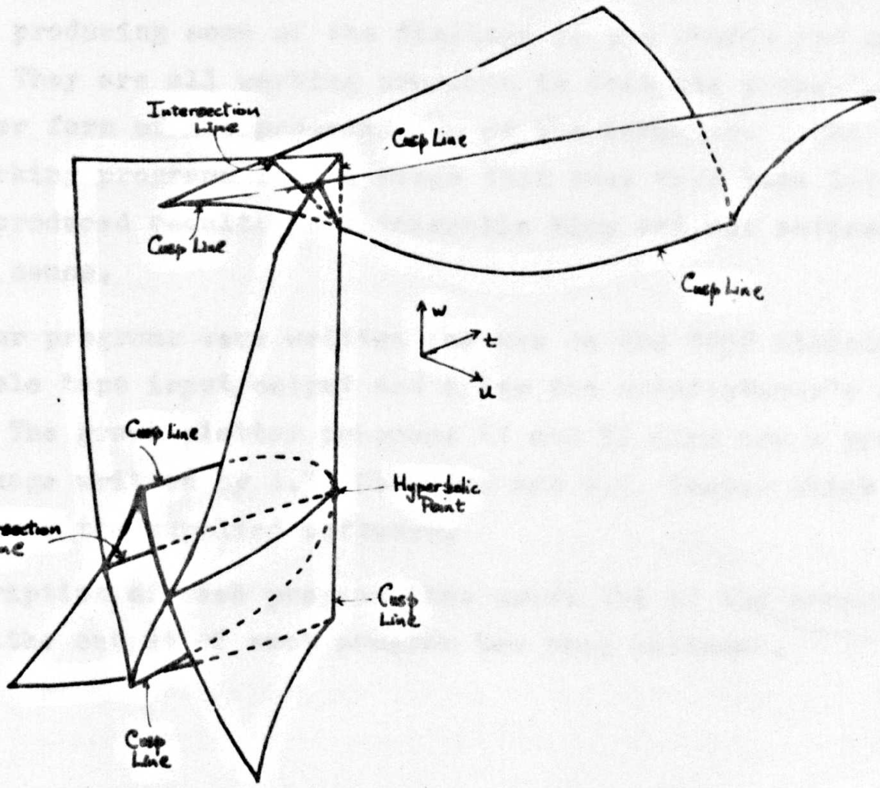

w
2 Min
1 Min
1 Min
t
Parabolic Point
1 Max + 1 Min
1 Max

FIG 15

V<0


Intersection Line
Cusp Line
Cusp Line
Cusp Line
w
t
u
Cusp Line
Hyperbolic Point
Intersection Line
Cusp Line
Cusp Line

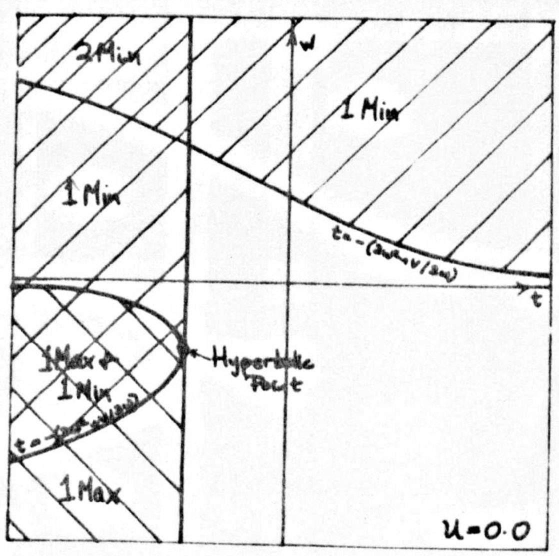

2Min
w
1 Min
1 Min
t
1Max + 1Min
Hyperbolic Point
1Max
u=0.0

FIG 16

APPENDIX II

General Note on the Computer Programs

The four programs of this appendix have been referred to in the text of Chapters II, III and Appendix I. Their results have been used directly in producing some of the findings of the thesis and are thus included. They are all working programs in that the presented form is a teleprinter form of the programs run on the computer. The programs are also working programs in the sense that they have been left in the state that produced results in a tolerable time and not refined in a pure programming sense.

All four programs were written for use on the 803B Elliott computer with five hole tape input/output and using the manufacturer's Algol compiler. The graph plotter programs (1 and 3) also use a precompiled plotter package written by A.E. Chantler and N.C. Bowyer which has many advantages over the supplied software.

A description of each program, the print out of the program and a sample from the output of each program has been included.

Program 1

Program Description

The notations for this description and the variable names of the program are those used in Appendix I.

The program reads as data a pair of values, representing w,t values. Using these values it first calculates suitable ranges for the parameter y and ranges for the u,v-graph it is to draw. With these ranges of u,v is sets up scaling factors for the graph and then draws suitable axes. Dividing up the y ranges into equal steps, or taking equal steps towards infinity, the program plots the corresponding u,v points relative to the previously drawn axes. The parameterless procedure CALC which plots a symmetric pair of points for a given y value checks that the u,v point is within the range of axes.

The program was quite quick to run, although considerable savings in time could have been made by restricting the picture to just the $u \geq 0$ half plane. Comparing with program 3 we see that the present program omits several features that could have been included. The graph itself could have been drawn rather than just the discrete points and more importantly some consideration of the distance between points given by successive y values could have been included. Further, it would save hand calculation checks if in the cases of the y range being infinite some method of continuing until the edge of the graph was reached had been included and some consideration was given to the position of cusps.

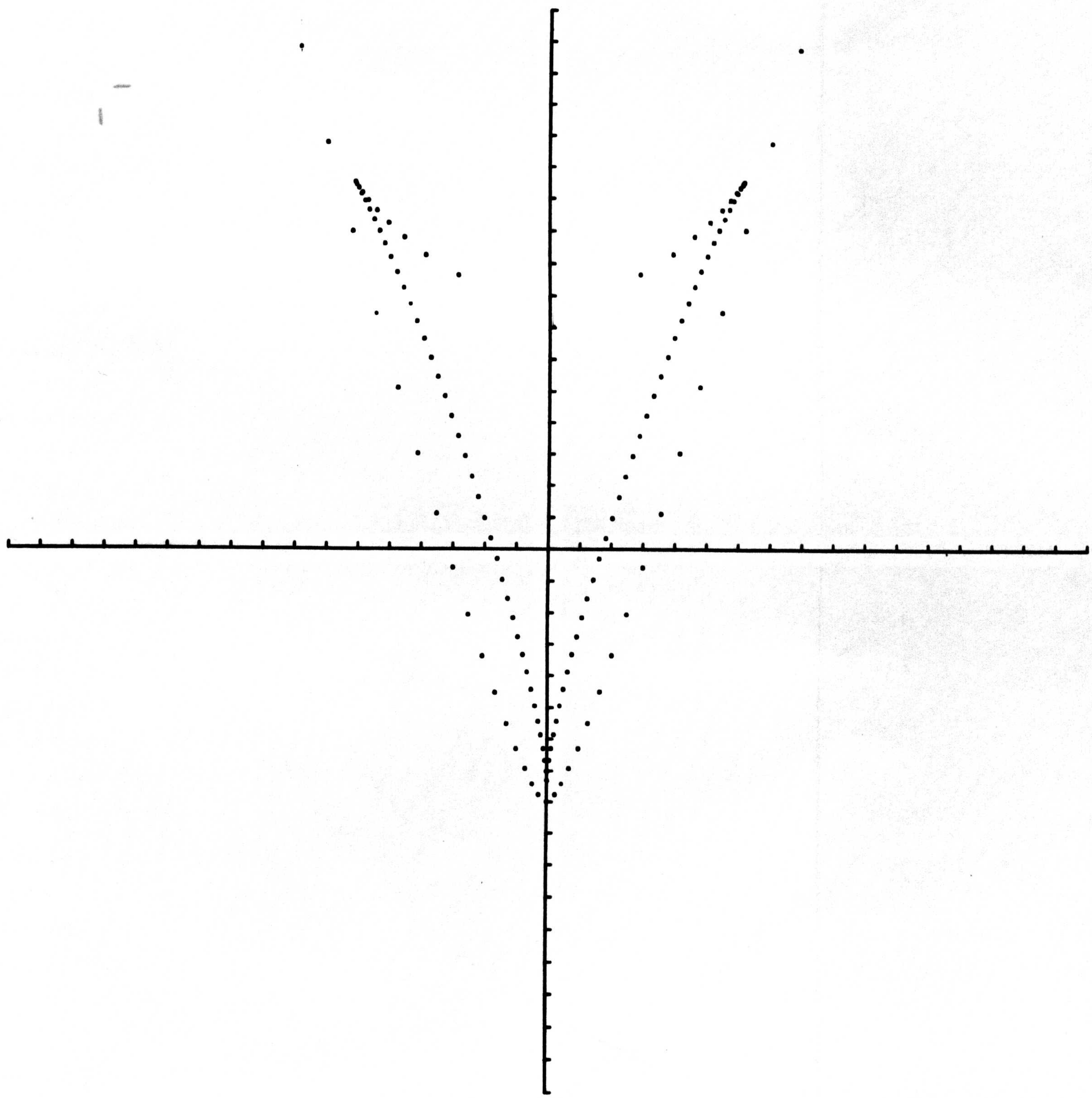
PROGRAM 1

PROGRAM 1 - 1

```
PARABOLIC UMBILIC - Y*Y*Y*Y STABILISER - GRAPHS
ANG 15071'
BEGIN REAL W,T,U,V,ROOT1,ROOT2,ROOT3,X,Y,OLDY,OLDU,OLDV,
 VTRNS1,VTRNS2,VTRNS3,FS,GRF,SC'
 INTEGER NUM'
 SWITCH SS:=S1'
 BOOLEAN AA'
 PROCEDURE CALC'
 BEGIN X:=SQRT((Y+W)*(6*Y*Y+T))'
  U:=2*X*(Y+W)'
  V:=X*X+4*Y**3+2*T*Y'
  IF ABS(U) LESS NUM AND ABS(V) LESS NUM THEN
  BEGIN MOVEPEN(U,V)'   CENCHARACTER(1)'
   MOVEPEN(-U,V)'   CENCHARACTER(1)'
  END'
 END OF PROCEDURE CALC'
 AA:=TRUE'
 READ T,W'
 VTRNS3:=ABS(-4*W*W*W-2*T*W)'
 IF T LESS 0 THEN
 BEGIN VTRNS1:=-4*T*SQRT(-T/6)/3'
  VTRNS2:=-VTRNS1'
  GRF:=IF VTRNS3 LESS VTRNS1 THEN 2*VTRNS1+1 ELSE 2*VTRNS3+1'
 END ELSE GRF:=2*VTRNS3+1'
 SC:=2000/(2*GRF)'
 NUM:=ENTIER(GRF)'
 SETORIGIN(1600,SC,SC,1)'
 AXES(1,1,NUM,NUM,NUM,NUM)'
 CENSET(1)'
 ROOT3:=-W'
 IF T LESS 0 THEN
 BEGIN ROOT1:=SQRT(-T/6)'
  ROOT2:=-ROOT1'
  IF ROOT3 LESSEQ ROOT1 THEN
  BEGIN FS:=0.05'
   OLDY:=ROOT1'
   FOR Y:=ROOT1 STEP FS UNTIL ROOT1+1.0001 DO CALC'
   FS:=1'
   FOR Y:=ROOT1+1 STEP FS UNTIL ROOT1+120.0001 DO CALC'
   IF ROOT3=ROOT2 THEN
   BEGIN Y:=ROOT3'
    CALC'
   END ELSE
   BEGIN FS:=(ROOT2-ROOT3)/50'
    OLDY:=ROOT3'
    FOR Y:=ROOT3 STEP FS UNTIL ROOT2 DO CALC'
   END
  END ELSE
  BEGIN FS:=(ROOT1-ROOT2)/50'
   OLDY:=ROOT2'
   FOR Y:=ROOT2 STEP FS UNTIL ROOT1 DO CALC'
S1: AA:=FALSE'
   FS:=0.05'
   FOR Y:=ROOT3 STEP FS UNTIL ROOT3+1.0001 DO CALC'
   FS:=1'
   FOR Y:=ROOT3+1 STEP FS UNTIL ROOT3+120.0001 DO CALC'
  END'
 END ELSE IF AA THEN GOTO S1'
 END'
END'
END'
```

SAMPLE OUTPUT - PROGRAM 1

T=-6.0
W=-0.75



Program 2

Program Description

This program is designed to give a numerical determination of the critical points and critical values of the polynomial

$$x^2y + y^4 + wx^2 + ty^2 - ux - vy . \qquad (1)$$

It relates specifically to the cases with $w = 1$, t being such that the region of u,v plane with 2 minima (cf. Appendix I, Fig. 11) not empty and u,v being within this region. The initial value of t is read as data and then calculations of the ranges of u,v values to be dealt with are made. This calculation of u,v limits requires the solution of a cubic equation using procedure SOLV. The u,v region is then scanned and for each set of values the 5 critical points of the function (1) are obtained by solving, using SOLV, the quintic equation

$$(y^3 + 2ty - v)\ (y + w)^2 + u^2/4 = 0$$

and then substituting the y values in various simple formula to get the x value, the critical value and the type of the critical point. When this has been completed for all the u,v values of the mesh given by a particular value of t, t is altered by a suitable amount and the u,v process repeated.

The procedure SOLV takes advantage of the fact that the cubic and quintic will have their full complement of real roots in the region of interest. It starts by using the procedure LAGRT to find the largest root. The procedure LAGRT itself uses Newton's iteratative method, (procedure NEWTR) starting at some point at which the polynomial and all its derivatives are positive. SOLV then uses the procedure DIVIDE to reduce the polynomial and continues to find all the roots. When we have reduced to a quadratic equation there is the procedure QUADRAT to get the final pair of roots.

It is very easy to alter the program to generalise the steps in the u,v scan and in the t scan. Further, it is not much more difficult to add in a possibility for all positive w, with the value as data. However, to generalise to all u,v,w,t and to get out the critical points and critical values is not quite so easy since these require us to allow the possible cases of cubics and quintics without all real roots.

Since the results of the program were required only to estimate the form of the Maxwell set in the 2 minima region the generalisations were not undertaken.

```
CATASTROPHE SET FOR PARABOLIC UMBILIC
ANG 15071'
BEGIN ARRAY Y,UR(1:3),RLRTS(1:5),POL1(0:3),POL2(0:5)'
 BOOLEAN RECT'
 INTEGER COUNT,I1,D'
 REAL T,JUMP,U,V,YCRIT,LWV,LWU,UPU,UPV,ATEM,VSTP,USTP,XVAL,
  TPIND,FVAL'
 SWITCH MSW:=INIT1,INIT2,BR,TNX,VNX,UNX,TYNX'



PROCEDURE NEWTR(FIRST,SQR,SIZ,ANS)'
 INTEGER SIZ'
 REAL FIRST,ANS'
 ARRAY SQR'
BEGIN INTEGER N1'
 SWITCH NS:=NS1'
 REAL Y,NEWY'
 REAL PROCEDURE FRACT(E,APP,FSIZ)'
  INTEGER FSIZ'
  REAL APP'
  ARRAY E'
 BEGIN INTEGER F1'
  REAL TEM1,TEM2'
  TEM1:=TEM2:=0'
  FOR F1:=FSIZ STEP -1 UNTIL 0 DO
   TEM1:=TEM1*APP+CHECKR(E(0,F1))'
  FOR F1:=FSIZ-1 STEP -1 UNTIL 0 DO
   TEM2:=TEM2*APP+CHECKR(E(1,F1))'
  IF ABS(TEM2) LESS 1.0@-10 THEN
  BEGIN PRINT ££L?NEWTON UNRELIABLE £U??'
   WAIT'
  END'
  FRACT:=CHECKR(TEM1)/CHECKR(TEM2)'
  CHECKS(£FRACT EXIT?)'
 END FRACT'
 CHECKS(£ENTER NWTR?)'
 N1:=1'
 Y:=FIRST'
 NS1:NEWY:=CHECKR(Y)-FRACT(SQR,Y,SIZ)'
 N1:=N1+1'
 IF N1 LESS 100 AND ABS(NEWY-Y) GR 1.0@-05 THEN
 BEGIN Y:=NEWY'
  GOTO NS1'
 END ELSE IF N1=100 THEN
  BEGIN PRINT ££L?SLOW CONVERGENCE OF NEWTON?'
   WAIT'
   Y:=NEWY'
   GOTO NS1'
  END'
 ANS:=NEWY'
 CHECKS(£NEWTR EXIT?)'
END OF NEWTR'
$
```

```
ROCEDURE DIFST(A,B,H)'
ARRAY A,B'
INTEGER H'
EGIN INTEGER DF1,DF2'
 CHECKS(£ENTER DIFST?)'
FOR DF1:=0 STEP 1 UNTIL H DO
  B(0,DF1):=CHECKR(A(DF1))'
FOR DF2:=1 STEP 1 UNTIL H DO
BEGIN
 FOR DF1:=0 STEP 1 UNTIL H-DF2 DO
  B(DF2,DF1):=CHECKR((DF1+1)*B(DF2-1,DF1+1))'
 FOR DF1:=H-DF2+1 STEP 1 UNTIL H DO
  B(DF2,DF1):=0'
END'
CHECKS(£DIFST EXIT?)'
ND OF DIFST'

ROCEDURE DIVIDE(PAR,PD,XMA,NPAR,REM)'
ARRAY PAR,NPAR'
REAL XMA,REM'
INTEGER PD'
EGIN INTEGER DV1'
 CHECKS(£ENTER DIVIDE?)'
NPAR(PD-1):=PAR(PD)'
FOR DV1:=PD-2 STEP -1 UNTIL 0 DO
 NPAR(DV1):=NPAR(DV1+1)*XMA+PAR(DV1+1)'
REM:=NPAR(0)*XMA+PAR(0)'
CHECKS(£DIVIDE EXIT?)'END OF DIVIDE'

ROCEDURE QUADRAT(A1,B1,C1,ROOT1,ROOT2)'
REAL A1,B1,C1,ROOT1,ROOT2'
EGIN REAL Q1'
SWITCH QJ:=QJ1,QJ2'
CHECKS(£ENTER QUADRAT?)'
Q1:=B1*B1-4*A1*C1'
IF Q1 LESS 0 THEN
BEGIN PRINT ££L?QUADRATIC WITH COMPLEX ROOTS?'
 WAIT'
GOTO QJ1'
END'
J1:IF A1=0 THEN BEGIN PRINT ££L?QUAD IS LINEAR?'
 IF B1=0 THEN
 BEGIN PRINT ££L?QUAD IS CONST?'
  WAIT'
  ROOT1:=ROOT2:=C1'
  GOTO QJ2'
 END'
 ROOT1:=ROOT2:=-C1/B1'
 GOTO QJ2'
END'
ROOT1:=(B1-SQRT(ABS(Q1)))/(A1*(-2))'
ROOT2:=(B1+SQRT(ABS(Q1)))/(-2*A1)'
J2:CHECKS(£QUADRAT EXIT?)'
ND'
```

PROGRAM 2.3.

```
EAL PROCEDURE LAGRT(POL,N)'
ARRAY POL'
INTEGER N'
EGIN ARRAY EXPOL(0:N,0:N),VLS(0:N)'
INTEGER L1'
SWITCH SS:=SW1,SW2'
REAL RT,X'
PROCEDURE EVDIF(X,A,R,B)'
 ARRAY A,B'
 INTEGER R'
 REAL X'
BEGIN INTEGER EV1,EV2'
 CHECKS(£ENTER EVDIF?)'
 FOR EV1:=0 STEP 1 UNTIL R DO
 BEGIN B(EV1):=0'
  FOR EV2:=R-EV1 STEP -1 UNTIL 0 DO
   B(EV1):=B(EV1)*X+A(EV1,EV2)'
 END'
 CHECKS(£EVDIF EXIT ?)'
END OF EVDIF'
CHECKS(£ENTER LAGRT?)'
DIFST(POL,EXPOL,N)'
X:=ABS(POL(0))'
SW1:EVDIF(X,EXPOL,N,VLS)'
L1:=0'
SW2:IF VLS(L1) GR 0 AND L1 LESS N THEN
BEGIN L1:=L1+1'
 GOTO SW2'
END ELSE IF L1 NOTEQ N THEN
 BEGIN X:=X+1'
  GOTO SW1'
 END'
NEWTR(X,EXPOL,N,RT)'
LAGRT:=RT'
CHECKS(£LAGRT EXIT?)'
ND OF LAGRT'
```

PROGRAM 2.4

PROGRAM 2 - 4

```
ROCEDURE SOLV(STR,DEG,ROUT)'
ARRAY STR,ROUT'
INTEGER DEG'
EGIN IF DEG=3 THEN
BEGIN ARRAY ALT(0:2)'
 REAL TST'
 CHECKS(£ENTER CUBIC SOLV?)'
 ROUT(1):=LAGRT(STR,DEG)'
 DIVIDE(STR,DEG,ROUT(1),ALT,TST)'
 IF ABS(TST) GR 1.0@-5 THEN PRINT ££L?LAGRT NOT ACC FOR CUBIC?'
 QUADRAT(ALT(2),ALT(1),ALT(0),ROUT(2),ROUT(3))'
 CHECKS(£CUBIC SOLV EXIT?)'
END ELSE IF DEG=5 THEN
 BEGIN ARRAY ALT1(0:4),ALT2(0:3),ALT3(0:2)'
  REAL TST'
  INTEGER S1'
  CHECKS(£ENTER QUINTIC SOLV?)'
  ROUT(1):=LAGRT(STR,DEG)'
  DIVIDE(STR,DEG,ROUT(1),ALT1,TST)'
  IF ABS(TST) GR 1.0@-5 THEN
   PRINT ££L?LAGRT NOT ACC FOR QUINTIC?'
  ROUT(2):=LAGRT(ALT1,4)'
  DIVIDE(ALT1,4,ROUT(2),ALT2,TST)'
  IF ABS(TST) GR 1.0@-05 THEN PRINT ££L?LAGRT NOT CURATE?'
  ROUT(3):=LAGRT(ALT2,3)'
  DIVIDE(ALT2,3,ROUT(3),ALT3,TST)'
  IF ABS(TST) GR 1.0@-05 THEN PRINT££L?STILL NOT ACC?'
  QUADRAT(ALT3(2),ALT3(1),ALT3(0),ROUT(4),ROUT(5))'
 END'
ND OF SOLV'
```

```
JUMP:=1'
READ T'

TNX:T:=T+JUMP'
WAIT'
RECT:=T LESS 0'
PRINT ££L?T VALUE = ?,SAMELINE,T,££L2??'
YCRIT:=-(SQRT(-T/10+0.04)+0.2)'
UPV:=((10*YCRIT+6)*YCRIT+3*T)*YCRIT+T'
IF RECT THEN LWV:=(IF 4*(T/3)*SQRT(-T/6) GR -2*(2+T)
 THEN 4*(T/3)*SQRT(-T/6) ELSE -2*(2+T))
 ELSE BEGIN YCRIT:=-(0.2-SQRT(-T610+0.04))'
  LWV:=((10*YCRIT+6)*YCRIT+3*T)*YCRIT+T'
 END'
POL1(1):=3*T'
POL1(2):=6'
POL1(3):=10'
V:=LWV'
VSTP:=(UPV-LWV)/6'
GOTO INIT1'
NX:POL1(0):=T-V'
SOLV(POL1,3,Y)'
COUNT:=1'
I1:=1'
R:ATEM:=(6*Y(I1)*Y(I1)+T)*(Y(I1)+1)'
IF ATEM GREQ 0 THEN
BEGIN UR(COUNT):=2*SQRT(ATEM*(Y(I1)+1)**2)'
 COUNT:=COUNT+1'
END'
I1:=I1+1'
IF I1 LESSEQ 3 THEN GOTO BR '
IF COUNT=4 THEN
BEGIN IF RECT THEN PRINT ££L?TOO MANY U VALUES?'
 IF UR(1) LESSEQ UR(2) THEN
 BEGIN IF UR(3) LESSEQ UR(1) THEN
  BEGIN LWU:=UR(3)'
   UPU:=UR(1)'
  END ELSE
  BEGIN LWU:=UR(1)'
   UPU:=IF UR(3) LESSEQ UR(2) THEN UR(3) ELSE UR(2)'
  END
 END ELSE IF UR(3) LESSEQ UR(2) THEN
  BEGIN LWU:=UR(3)'
   UPU:=UR(2)'
  END ELSE
  BEGIN LWU:=UR(2)'
   UPU:=IF UR(3) LESSEQ UR(1) THEN UR(3) ELSE UR(1)'
  END'
END ELSE
BEGIN IF COUNT LESS 3 OR NOT RECT THEN PRINT
££L?TOOFEW VALUES OF U £U??'
LWU:=0'
UPU:=IF UR(1) LESSEQ UR(2) THEN UR(1) ELSE UR(2)'
END'
U:=LWU'
USTP:=(UPU-LWU)/6'
GOTO INIT2'
```

PROGRAM 2·6

```
X:PRINT ££L?U VALUE = ?,SAMELINE,U,££S10?V VALUE = ?,
V,£
  X            Y        TYPE            VALUE?'
POL2(5):=1'
POL2(4):=2'
POL2(3):=1+T/2'
POL2(2):=T-V/4'
POL2(1):=(T-V)/21
POL2(0):=U*U/16-V/4'
SOLV(POL2,5,RLRTS)'
D:=1'
YNX:XVAL:=U/(2*(RLRTS(D)+1))'
PRINT XVAL,SAMELINE,£ ?,RLRTS(D)'
TPIND:=(RLRTS(D)**2*6+T)*(RLRTS(D)+1)-XVAL**2'
IF TPIND=0 THENNHTZIUKKMUKDEGEN ?
ELSE IF TPIND GR 0 THEN
BEGIN IF RLRTS(D) GR -1.0 THEN PRINT £  MIN  ?
 ELSE PRINT £  MAX  ?
END ELSE PRINT £ SADDLE?'
FVAL:=XVAL*XVAL*RLRTS(D)+RLRTS(D)**4+XVAL**2+T*RLRTS(D)**2-
 U*XVAL -V*RLRTS(D)'
PRINT SAMELINE ,£ ?,FVAL'
D:=D+1'
IF D LESSEQ 5 THEN GOTO TYNX'
IT2:U:=U+(IF USTP NOTEQ 0 THEN USTP ELSE 1)'
IF U LESS UPU THEN GOTO UNX'
IT1:V:=V+(IF VSTP NOTEQ 0 THEN VSTP ELSE 1)'
IF V LESS UPV THENDGOTO VNX'
IF T LESS 0 THEN GOTO TNX
ELSE BEGIN JUMP:=0.1'
 IF T LESS 0.4 THEN GOTO TNX'
END'
D'
```

SAMPLE OUTPUT – PROGRAM 2.

```
T VALUE =  .10000000


U VALUE =  .71098408                V VALUE =  .13071797
     X            Y          TYPE             VALUE
 .31885315   .11490835   MIN     -.12687561
 .38206679 -.06955524 SADDLE -.12622238
 .42320695 -.16000424   MIN     -.12631570
 1.3876621 -.74381944 SADDLE -.03464074
-2.5117888 -1.1415294 SADDLE  2.8704904
U VALUE =  .71441177                V VALUE =  .13071797
     X            Y          TYPE             VALUE
 .32211375   .10894331   MIN     -.12797405
 .37506072 -.04760518 SADDLE -.12751929
 .43439389 -.17769127   MIN     -.12778626
 1.3820463 -.74153841 SADDLE -.03938761
-2.5136149 -1.1421084 SADDLE  2.8791032
U VALUE =  .71783947                V VALUE =  .13071797
     X            Y          TYPE             VALUE
 .32571273   .10195183   MIN     -.12908420
 .36954498 -.02875223 SADDLE -.12879520
 .44381568 -.19128649   MIN     -.12929171
 1.3763652 -.73922638 SADDLE -.04411513
-2.5154388 -1.1426867 SADDLE  2.8877222
U VALUE =  .72126717                V VALUE =  .13071797
     X            Y          TYPE             VALUE
 .32984897   .09332943   MIN     -.13020754
 .36441215 -.01036893 SADDLE -.13005309
 .45238328 -.20281407   MIN     -.13082783
 1.3706162 -.73688215 SADDLE -.04882305
-2.5172606 -1.1432643 SADDLE  2.8963475
U VALUE =  .72469486                V VALUE =  .13071797
     X            Y          TYPE             VALUE
 .33504259   .08149664   MIN     -.13134661
 .35879521   .00990040 SADDLE -.13129287
 .46044613 -.21305142   MIN     -.13239240
 1.3647967 -.73450447 SADDLE -.05351116
-2.5190804 -1.1438412 SADDLE  2.9049790
U VALUE =  .78542517                V VALUE =  .16535898
     X            Y          TYPE             VALUE
 .33644110   .16725509   MIN     -.15620180
 .43171161 -.09033582 SADDLE -.15371809
 .51038793 -.23056060   MIN     -.15416888
 1.2781917 -.69275926 SADDLE -.10909770
-2.5567324 -1.1535994 SADDLE  3.0989044
U VALUE =  .79414823                V VALUE =  .16535898
     X            Y          TYPE             VALUE
 .34249765   .15934845   MIN     -.15916270
 .42171311 -.05842597 SADDLE -.15743709
 .53739790 -.26111711   MIN     -.15874171
 1.2596513 -.68477458 SADDLE -.12016725
-2.5612594 -1.1550308 SADDLE  3.1212267
U VALUE =  .80287129                V VALUE =  .16535898
     X            Y          TYPE             VALUE
 .34906742   .15002323   MIN     -.16217853
 .41443304 -.03136187 SADDLE -.16408290
```

Program 3

Program Description

The program as originally drafted was to draw (u,v) sections for the bifurcation set in the 6 dimensional unfolding

$$(x^4 + y^4)/4 + \alpha x^2 y + \beta xy^2 + wx^2 + ty^2 - ux - vy .$$

It was, however, only developed far enough to deal with the cases in which $\alpha\beta = 0$, $\alpha + \beta \neq 0$. Two modes of use for the program are possible :

(i) Read integer 0 followed by a set of values for α,w,t and output the particular u,v-graph.

(ii) Read integer $\neq 0$, followed by a pair of values of α,β which if within the compass of the program give an output of u,v-graphs for a range of w,t values.

The choice of w,t values will give all different types of u,v-graph behaviour deducible from analogues of Figs. 1 and 2 of Chapter III.

Most of the work of the program is done by procedure GRAPH which has as parameters values for α,w,t and outputs the corresponding u,v-graph. This procedure also uses values of global variables giving y and v values of points of the graph on n = 0 which are calculated in the main program. Inside the procedure GRAPH the values of the global variables mentioned are used in a calculation that similates consideration of a particular diagram like those of Fig. 1 and 2 of Chapter III. This produces axes and then a series of y ranges which give branches of the u,v-graph and each branch is drawn by a sub-procedure DRBRCH. This latter procedure has as input parameters Boolean variables giving ;y range of branch finite or not ;v range of branch finite or not ; branch meets the axis or not : as well as the beginning and end (or somewhere towards end if y range infinite) of the y range for the branch. Using this information the procedure DRBRCH draws the successive linear sections of the (u,v)-graph between successive values chosen for y. These values are chosen with due regard for the distance between the successive (u,v) points produced, not too little or too much distance being allowed. Further the procedure checks that the points are within the range of the axes and will allow a branch to go off and then come back onto the picture being drawn or even start outside the axes and only produce points within the range of the axes when it gets there. Other sophistications speed up the symmetric picture on the plotter.

Even with this fairly sophisticated program for drawing the (u,v)-graphs difficulties were encountered with some branches in the neighbourhood of points 'Sp' of Figs. 1, 2 of Chapter III. To get results of Chapter III, particularly the transition of Fig. 3, it was necessary to use Program 4. It would be possible to refine the present program to deal with this case but this has not been done since sufficient results had been obtained more simply.

PROGRAM 3.11

```
NWINDING X**4+Y**4 FIRST STAGE
NG 15071'

EGIN REAL ALPHA,Y,BETA,T,W,FACT,X2'
INTEGER I,CONST,K,COUNT,P'
ARRAY YLEV,VLEV(1:7),LEV(1:5)'
SWITCH SS:=SW1,SW2,SW3,SW4,SW5,SW6,SW7'
PROCEDURE GRAPH(A,B,C)'
 REAL A,B,C'
BEGIN REAL TY,TYEND,U,V,VMAX'
ROCEDURE DRBRCH(YFIN,VFIN,MEAX,Y,YEND)'
REAL Y,YEND'
BOOLEAN YFIN,VFIN,MEAX'
EGIN ARRAY STU,STV(1:25)'
INTEGER T,J,HC,DC'
REAL DIST,TU,TV,YSTP,OLDU,OLDV,X2,PAR'
BOOLEAN NWBR,AGAIN'
SWITCH DR:=DR1,DR2,DR3,DR4'
AGAIN:=1=0'
R4:NWBR:=1=0'
PAR:=Y'
IF MEAX THEN
BEGIN STU(1):=OLDU:=0'
 STV(1):=OLDV:=PAR*(PAR*PAR+2*C)'
 IF PAR=YLEV(2) OR PAR=YLEV(3) THEN
  BEGIN X2:=-2*(A*Y+W)'
  STV(1):=STV(1)+A*X2'
  OLDV:=STV(1)'
 END'
 MOVEPEN(OLDU,OLDV)'
 IF Y=YEND THEN BEGIN CENCHARACTER(1)' GOTO DR2' END'
 NWBR:=1=1'
END'
J:=IF MEAX THEN 2 ELSE 1'
HC:=DC:=0'
YSTP:=(YEND-Y)/50'
R1:PAR:=PAR+YSTP'
IF (IF YSTP GR 0 THEN PAR GR YEND ELSE PAR LESS YEND) THEN
BEGIN IF  YFIN THEN
 BEGIN MOVEPEN(STU(1),STV(1))'
  FOR T:=2 STEP 1 UNTIL J-1 DO DRAWLINE(STU(T),STV(T))'
  GOTO DR2'
 END ELSE
 BEGIN YEND:=YEND+YSTP'
  GOTO DR1'
 END'
```

```
END ELSE
BEGIN X2:=-2*(A*PAR+B)*(PAR*PAR+2*C/3)/(3*(PAR*PAR+2*C/3-4*A*A/9))'
U:=SQRT(X2)*(2*(A*PAR+B)+X2)'
V:=A*X2+PAR*(PAR*PAR+2*C)'
IF NWBR THEN
BEGIN DIST:=(ABS(U-OLDU)+ABS(V-OLDV))*750/VMAX'
 IF DIST LESS 4 THEN
 BEGIN DC:=DC+1'
  IF DC LESS 6 THEN
  BEGIN PAR:=PAR-YSTP'
   YSTP:=YSTP*2'
   GOTO DR1'
  END'
 END ELSE
 IF DIST GR 200 THEN
 BEGIN HC:=HC+1'
  IF HC LESS 6 THEN
  BEGIN PAR:=PAR-YSTP'
   YSTP:=YSTP/2'
   GOTO DR1'
  END'
 END'
END'
IF ABS(U)-2*VMAX GR 0 OR ABS(V)-2*VMAX GR 0 THEN
BEGIN IF NWBR THEN
 BEGIN NWBR:=1=0'
  MOVEPEN(STU(1),STV(1))'
  FOR T:=2 STEP 1 UNTIL J-1 DO DRAWLINE(STU(T),STV(T))'
   IF ABS(STU(J-1)) LESS VMAX AND ABS(STV(J-1)) LESS VMAX THEN
  BEGIN TU:=STU(J-1)+(STV(J-1)-2*VMAX)*(U+STU(J-1))/(V-STV(J-1))'
   TV:=2*VMAX'
   IF ABS(TU) GR 2*VMAX THEN
   BEGIN TU:=2*VMAX'
    TV:=STV(J-1)+(STU(J-1)-2*VMAX)*(V-STV(J-1))/(U+STU(J-1))'
   END'
   DRAWLINE(TU,TV)'
   MOVEPEN(-TU,TV)'
   DRAWLINE(-STU(J-1),STV(J-1))'
  END'

  IF VFIN THEN
  BEGIN DR3:OLDU:=U'
   OLDV:=V'     GOTO DR1'
  END ELSE GOTO DR2'
 END ELSE GOTO DR3'
END ELSE
BEGIN IF NWBR THEN
 BEGIN DRAWLINE(U,V)'
  STU(J):=-U'
  STV(J):=V'
  IF J=25 THEN
  BEGIN MOVEPEN(STU(1),STV(1))'
   FOR T:=2 STEP 1 UNTIL 25 DO DRAWLINE(STU(T),STV(T))'
   MOVEPEN(U,V)'
   STU(1):=-U'
   STV(1):=V'
   J:=1'
  END'
```

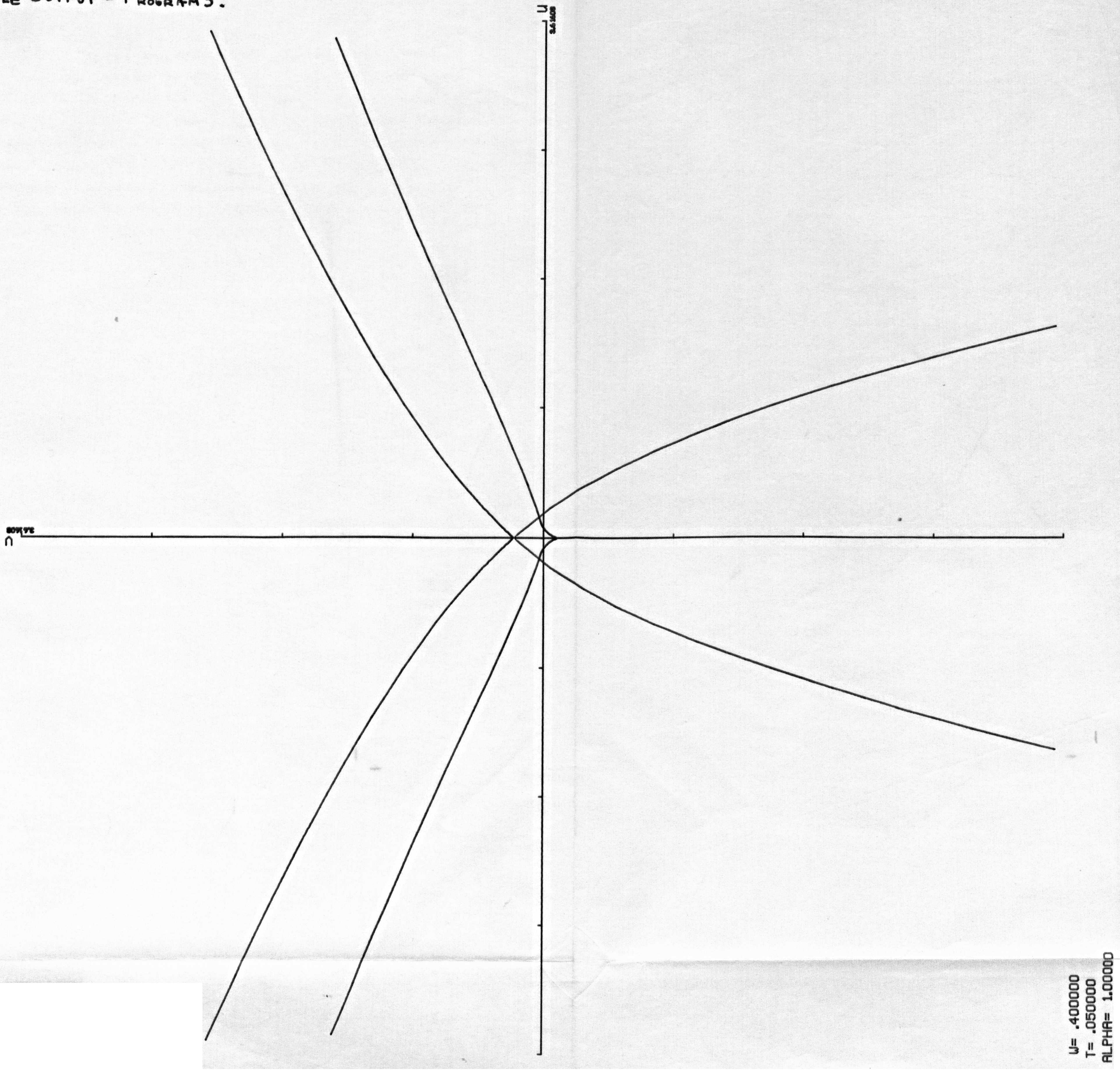
PROGRAM 3. 4

```
 DRBRCH(TRUE,TRUE,TRUE,TY,YLEV(3))'
END ELSE
IF COUNT=4 OR COUNT=5 THEN
BEGIN IF YLEV(1) LESSEQ YLEV(5) THEN
 DRBRCH(TRUE,NOT (YLEV(1)=YLEV(5) AND COUNT=4),TRUE,YLEV(3),YLEV(1))'
END ELSE
BEGIN IF YLEV(1) LESS YLEV(7) THEN
 BEGIN TY:=IF YLEV(1) GREQ YLEV(5) THEN YLEV(7) ELSE YLEV(3)'
  DRBRCH(TRUE,TRUE,TRUE,TY,YLEV(1))'
 END ELSE
 IF COUNT=7 THEN
 BEGIN TY:=IF YLEV(1) GR YLEV(6) THEN YLEV(6) ELSE YLEV(1)'
  DRBRCH(TRUE,TRUE,TRUE,TY,YLEV(7))'
 END'
END'
END'
F COUNT GR 3 THEN
EGIN IF YLEV(1) GREQ YLEV(5) THEN
BEGIN DRBRCH(TRUE,FALSE,TRUE,YLEV(3),YLEV(5))'
 IF YLEV(1) GREQ (IF COUNT LESS 6 THEN YLEV(5) ELSE YLEV(6)) THEN
 BEGIN TY:=IF YLEV(1) GR YLEV(2) THEN YLEV(2) ELSE YLEV(1)'
  DRBRCH(TRUE,FALSE,TRUE,TY,YLEV(4))'
 END'
ND'
IF COUNT=5 AND YLEV(1) LESS YLEV(5) THEN
 DRBRCH(TRUE,FALSE,FALSE,YLEV(4),YLEV(5))'
IF COUNT GR 5 THEN
EGIN IF YLEV(1) LESS YLEV(6) THEN
 DRBRCH(TRUE,FALSE,TRUE,YLEV(6),YLEV(4))'
 IF YLEV(1) LESS YLEV(5) THEN
  DRBRCH(TRUE,FALSE,TRUE,YLEV(7),YLEV(5))'
ND'
D'
VLEV(2):=VLEV(2)+2*A*B'
VLEV(3):=VLEV(3)+2*A*B'
ELLIOTT(0,6,0,0,7,0,0)'
ELLIOTT(0,3,CONST,0,2,0,K)'
ELLIOTT(7,3,4,1,4,3,1)'
IF K NOTEQ 0 THEN
BEGIN WAIT'
 DUMP'
END'
MOVEPEN(-2*VMAX,5*VMAX)'
ND OF GRAPH'
```

PROGRAM 3.5

```
READ I'
IF I=0 THEN
BEGIN READ ALPHA,W,T'
  GOTO SW6'
END'
READ ALPHA,BETA'
IF ALPHA*BETA NOTEQ 0 THEN GOTO SW1'
IF BETA NOTEQ 0 THEN GOTO SW2'
SW7: CONST:=7'
FACT:=ALPHA**2/6'
T:=-2*FACT'
SW6:COUNT:=1'
CONST:=7'
IF T LESSEQ 6*FACT THEN
BEGIN COUNT:=COUNT+1'
 YLEV(2):=SQRT(-2*T/3+4*FACT)'
 VLEV(2):=YLEV(2)*(YLEV(2)*YLEV(2)+2*(T-6*FACT))'
 YLEV(3):=-YLEV(2)'
 VLEV(3):=-VLEV(2)'
 IF T LESS 6*FACT THEN COUNT:=COUNT+1'
END ELSE GOTO SW3'
IF T LESSEQ 4*FACT THEN
BEGIN COUNT:=COUNT+1'
 YLEV(4):=SQRT(-2*T/3+8*FACT/3)'
 YLEV(5):=-YLEV(4)'
 IF T LESS 4*FACT THEN COUNT:=COUNT+1'
END ELSE GOTO SW3'
IF T LESSEQ 0 THEN
BEGIN COUNT:=COUNT+1'
 YLEV(6):=SQRT(-2*T/3)'
 VLEV(6):=YLEV(6)*(YLEV(6)*YLEV(6)+2*T)'
 YLEV(7):=-YLEV(6)'
 VLEV(7):=-VLEV(6)'
 IF T LESS 0 THEN COUNT:=COUNT+1'
END'
SW3:IF I=0 THEN BEGIN GRAPH(ALPHA,W,T)' STOP' END'
P:=2*COUNT+1'
LEV(1):=7*FACT'
LEV(2):=IF COUNT GR 2 THEN ALPHA*YLEV(2) ELSE 0'
LEV(3):=IF COUNT GR 4 THEN ALPHA*YLEV(4) ELSE 0'
LEV(4):=IF COUNT GR 6 THEN ALPHA*YLEV(6) ELSE 0'
LEV(5):=0'
SW4:W:=LEV(1)'
GRAPH(ALPHA,W,T)'
W:=-LEV(1)'
GRAPH(ALPHA,W,T)'
P:=P-2'
IF P=1 THEN GOTO SW5'
FOR I:=1 STEP 1 UNTIL 4 DO LEV(I):=LEV(I+1)'
GOTO SW4'
SW5:GRAPH(ALPHA,0,T)'
T:=T+FACT'
IF T LESSEQ 8*FACT THEN GOTO SW6'
STOP'
SW1:SW2:PRINT ££L? THIS SECTION NOT READY ?'
END'
END'
```

Sample Output - Program 3.



Program 4

Due to difficulties with certain w,t values with the u,v-graphs of Program 3, it was found necessary to do some extra calculations. Program 4 simply reads in the values of w,t and a range of values of the parameter y and prints out real finite values of ;u;v; sixth order polynomial to be solved for cusps ; dv/du for equal y intervals. This gives useful information in cases where the graphs of Program 3 and the analysis of Chapter III is not definite. In particular this program gave evidence to support to transitions illustrated in Fig. 3 of Chapter III, where the change in parameter giving large changes in u,v were too small to be detected by Program 3.

PROGRAM 4.1

```
INT OUT OF U,V VALUES
 15071'

IN REAL W,T,TEMP1,TEMP2,TEMP3,Y,UPPER,LOWER,GRAD,U,V,CUSP'
OLEAN EXTRA1,EXTRA2'
ITCH SS:=S1'
AD W,T,UPPER,LOWER'
INT ££L?W=  ?,SAMELINE,W,£  T=  ?,T'
INT ££LS5?Y£S9?U£S9?V£S8?CUSP£S6?GRAD  COMMENTS?'
=UPPER'
1:EXTRA1:=EXTRA2:=1=0'
PRINT Y'
TEMP1:=Y*Y+2.0*(T-2.0/3.0)/3.0'
TEMP2:=Y*Y+2.0*T/3.0'
IF ABS(TEMP1) LESS 0.000001 THEN
BEGIN EXTRA1:=1=1'
 U:=V:=GRAD:=0.0'
END ELSE
BEGIN TEMP3:=-2.0*(Y+W)*TEMP2/(3.0*TEMP1)'
 IF TEMP3 LESS 0.0 THEN
 BEGIN EXTRA2:=1=1'
  U:=GRAD:=0.0'
  V:=Y**3+2.0*T*Y+TEMP3'
 END ELSE
 BEGIN U:=SQRT(TEMP3)*(2.0*(Y+W)+TEMP3)'
  V:=Y**3+2.0*T*Y+TEMP3'
   GRAD:=-9.0*TEMP1*SQRT(TEMP3)/((Y+W)*4.0)'
 END'
END'
CUSP:=((Y*Y+2.0*(T-5.0/9.0))*Y*Y+4.0*(T*T-10.0*T/9.0+10.0/27.0)/3.0)
 *Y*Y+16.0*W*Y/81.0+8.0*T*(T-1.0)*(T-2.0/3.0)/27.0'
PRINT SAMELINE,U,£ ?,V,£ ?,CUSP,£ ?,GRAD'
IF EXTRA1 THEN PRINT ££S5?U,V LARGE?'
IF EXTRA2 THEN PRINT ££S5?U IMAG?'
=Y-(UPPER-LOWER)/100.0'
F Y LESSEQ LOWER THEN STOP ELSE GOTO S1'
D'
```

Sample Output - Program 4

= .83000000 T= -.33333333

Y	U	V	CUSP	GRAD	COMMENTS
.50000000	.08170619	.22300000	-.02519805	.34405119	
.50300000	.08537100	.22429462	-.02439202	.36254814	
.50600000	.08879391	.22556584	-.02359685	.38030929	
.50900000	.09200276	.22681354	-.02281270	.39743333	
.51200000	.09502005	.22803758	-.02203972	.41399938	
.51499999	.09786426	.22923783	-.02127807	.43007190	
.51799999	.10055086	.23041416	-.02052791	.44570430	
.52099999	.10309292	.23156644	-.01978938	.46094128	
.52399999	.10550162	.23269453	-.01906263	.47582079	
.52699999	.10778663	.23379833	-.01834779	.49037526	
.52999999	.10995637	.23487771	-.01764502	.50463271	
.53299999	.11201825	.23593254	-.01695442	.51861750	
.53599999	.11397882	.23696272	-.01627613	.53235098	
.53899998	.11584391	.23796812	-.01561028	.54585191	
.54199998	.11761875	.23894864	-.01495697	.55913696	
.54499998	.11930804	.23990417	-.01431632	.57222091	
.54799998	.12091603	.24083461	-.01368843	.58511701	
.55099998	.12244660	.24173985	-.01307340	.59783715	
.55399998	.12390328	.24261979	-.01247134	.61039201	
.55699998	.12528930	.24347435	-.01188232	.62279126	
.55999997	.12660765	.24430343	-.01130644	.63504367	
.56299997	.12786106	.24510694	-.01074378	.64715719	
.56599997	.12905208	.24588481	-.01019440	.65913906	
.56899997	.13018308	.24663695	-.00965837	.67099589	
.57199997	.13125626	.24736330	-.00913576	.68273372	
.57499997	.13227367	.24806379	-.00862662	.69435808	
.57799997	.13323724	.24873836	-.00813100	.70587403	
.58099996	.13414877	.24938695	-.00764894	.71728622	
.58399996	.13500997	.25000951	-.00718048	.72859891	
.58699996	.13582246	.25060600	-.00672566	.73981602	
.58999996	.13658775	.25117639	-.00628449	.75094111	
.59299996	.13730728	.25172066	-.00585700	.76197751	
.59599996	.13798242	.25223876	-.00544319	.77292823	
.59899996	.13861450	.25273071	-.00504306	.78379602	
.60199995	.13920475	.25319650	-.00465663	.79458341	
.60499995	.13975437	.25363613	-.00428386	.80529271	
.60799995	.14026451	.25404962	-.00392476	.81592599	
.61099995	.14073629	.25443702	-.00357929	.82648515	
.61399995	.14117075	.25479834	-.00324743	.83697189	
.61699995	.14156895	.25513367	-.00292913	.84738773	
.61999995	.14193187	.25544305	-.00262434	.85773399	
.62299995	.14226049	.25572659	-.00233303	.86801185	
.62599994	.14255577	.25598437	-.00205511	.87822232	
.62899994	.14281862	.25621652	-.00179052	.88836622	
.63199994	.14304997	.25642318	-.00153919	.89844425	
.63499994	.14325070	.25660451	-.00130103	.90845692	
.63799994	.14342170	.25676068	-.00107594	.91840461	
.64099994	.14356364	.25689191	-.00086382	.92828749	
.64399994	.14367800	.25699842	-.00066457	.93810563	
.64699993	.14376504	.25708047	-.00047805	.94785891	
.64999993	.14382583	.25713837	-.00030416	.95754701	
.65299993	.14386124	.25717242	-.00014274	.96716949	
.65599993	.14387214	.25718300	.00000635	.97672571	
.65899993	.14385944	.25717052	.00014325	.98621484	
.66199993	.14382403	.25713541	.00026815	.99563582	
.66499993	.14376684	.25707819	.00038120	1.0049875	
.66799992	.14368882	.25699940	.00048259	1.0142683	
.67099992	.14359093	.25689965	.00057251	1.0234766	
.67399992	.14347420	.25677962	.00065117	1.0326105	
.67699992	.14333966	.25664008	.00071877	1.0416677	